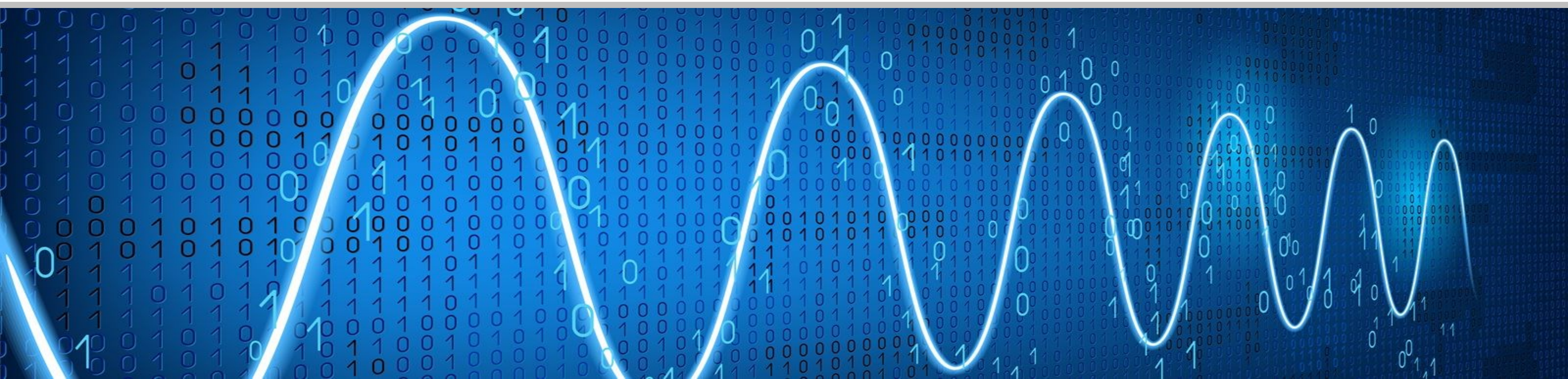
Digital Signal Processing

Lab 06: Analyzing Discrete-Time Systems

Abdallah El Ghamry



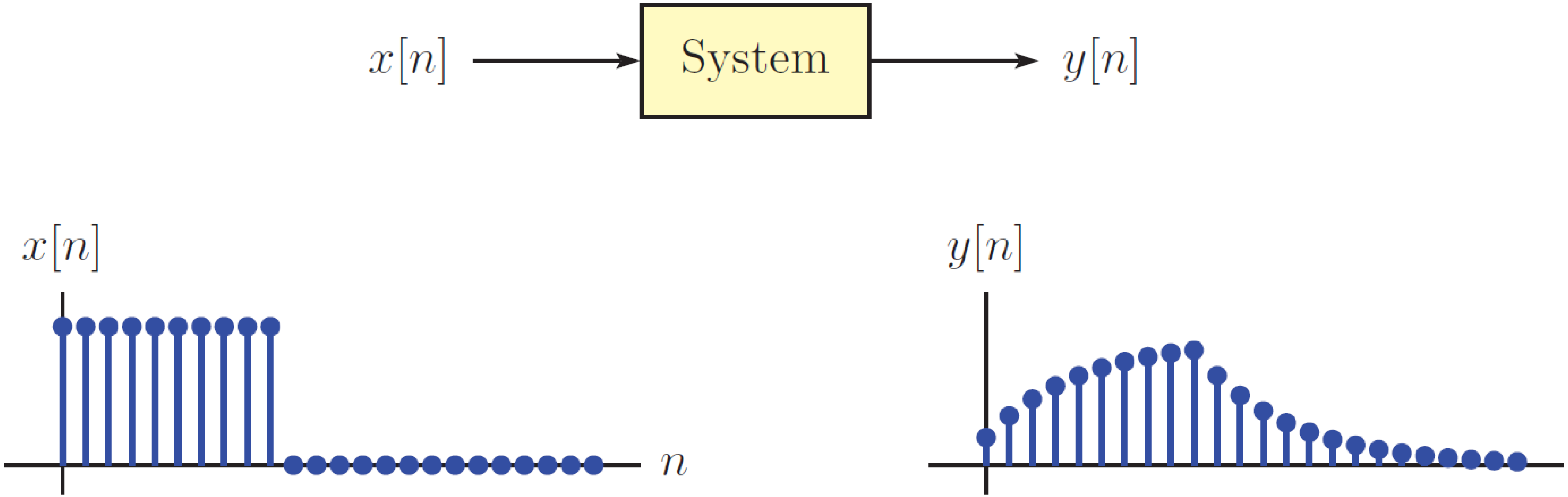
Analyzing Discrete-Time Systems in the Time Domain

The purpose of this lab is to

- Develop the notion of a discrete-time system.
- Discuss the concepts of linearity and time invariance.
- Learn how to compute the output signal for a linear and time-invariant system using convolution.
- Understand the graphical interpretation of the steps involved in carrying out the convolution operation.
- Learn the concepts of causality and stability.

Discrete-Time System

- A discrete-time system is a mathematical formula, method or algorithm that defines a cause-effect relationship between a set of discrete-time input signals and a set of discrete-time output signals.



Discrete-Time System

- The input-output relationship of a discrete-time system may be expressed in the form

$$y[n] = Sys\{x[n]\}$$

- A system that simply multiplies its input signal by a constant gain factor K

$$y[n] = K\,x[n]$$

- A system that delays its input signal by m samples

$$y[n] = x[n-m]$$

- A system that produces an output signal proportional to the square of the input signal

$$y[n] = K\,[x[n]]^2$$

Linearity

- Linearity property will be very important as we analyze and design discrete-time systems.
- Conditions for linearity of a discrete-time system are:

$$Sys\{x_1[n] + x_2[n]\} = Sys\{x_1[n]\} + Sys\{x_2[n]\}$$

$$Sys\{\alpha_1 x_1[n]\} = \alpha_1 Sys\{x_1[n]\}$$

Linearity

- The additivity rule can be stated as follows:

  The response of a linear system to the sum of two signals is the same as the sum of individual responses to each of the two input signals.

$$Sys\{x_1[n] + x_2[n]\} = Sys\{x_1[n]\} + Sys\{x_2[n]\}$$

- The homogeneity rule can be stated as follows:

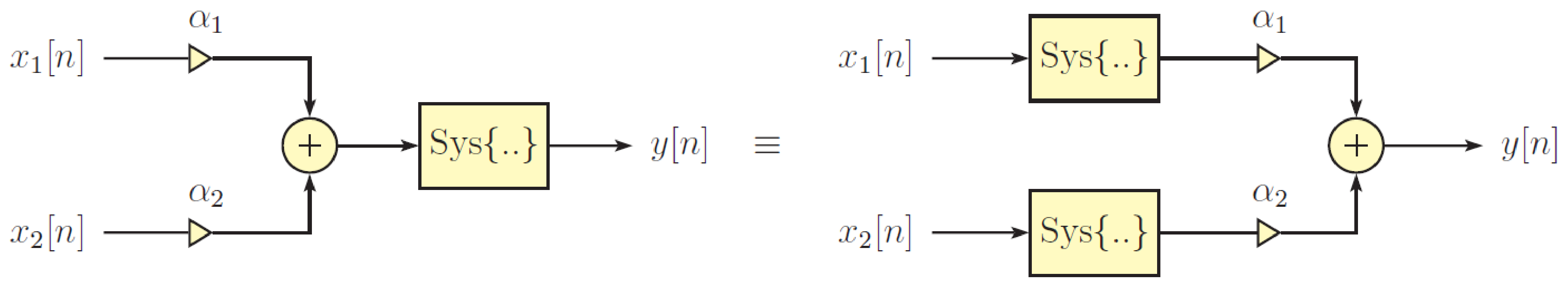
  Scaling the input signal of a linear system by a constant gain factor causes the output signal to be scaled with the same gain factor.

$$Sys\{\alpha_1 x_1[n]\} = \alpha_1 Sys\{x_1[n]\}$$

Linearity: Superposition Principle

- The two criteria can be combined into one equation which is referred to as the superposition principle.
- The response of the system to a weighted sum of any two input signals is equal to the same weighted sum of the individual responses of the system to each of the two input signals.

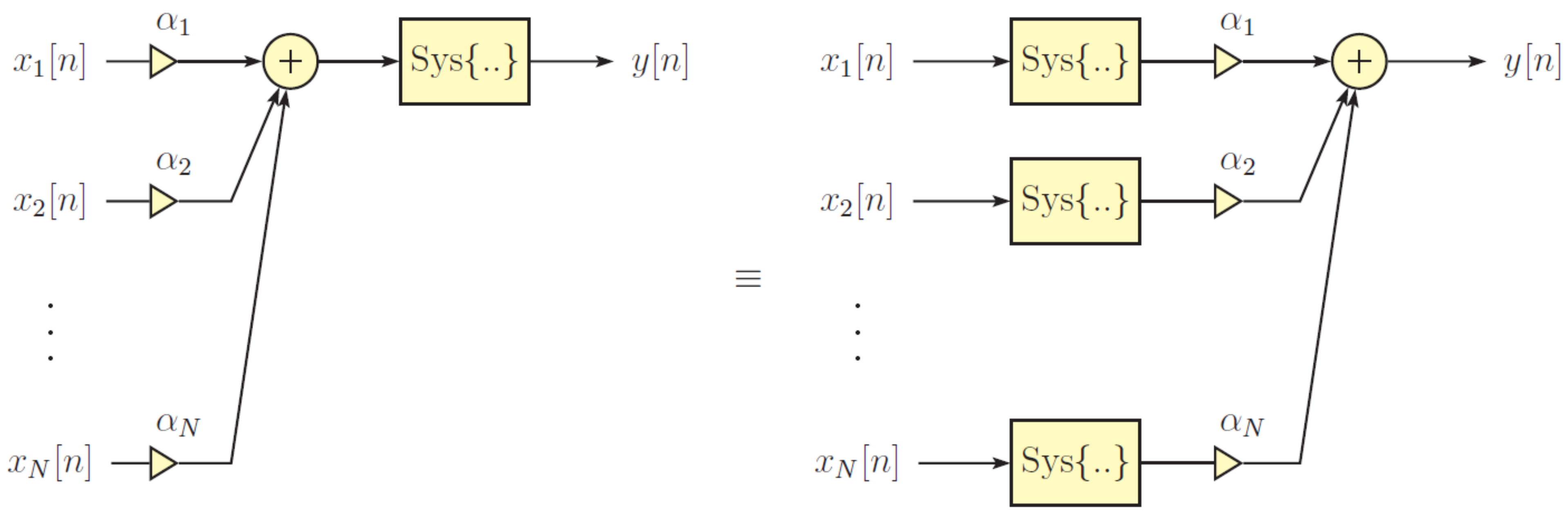
$$Sys\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\} = \alpha_1\, Sys\{x_1[n]\} + \alpha_2\, Sys\{x_2[n]\}$$



Linearity: Superposition Principle

- A generalization of the principle of superposition for the weighted sum of N discrete-time signals is expressed as

$$y[n] = \text{Sys}\left\{ \sum_{i=1}^{N} \alpha_i \, x_i[n] \right\} = \sum_{i=1}^{N} \alpha_i \, y_i[n]$$



Example 3.1

For each of the discrete-time systems described below, determine whether the system is linear or not:

a. $y[n] = 3\,x[n] + 2x[n-1]$

b. $y[n] = 3\,x[n] + 2\,x[n+1]\,x[n-1]$

c. $y[n] = a^{-n}\,x[n]$

Example 3.1 (a) – Solution

a. In order to test the linearity of the system we will think of its responses to the two discrete-time signals $x_1[n]$ and $x_2[n]$ as

$$y_1[n] = \text{Sys}\{x_1[n]\} = 3\,x_1[n] + 2\,x_1[n-1]$$

and

$$y_2[n] = \text{Sys}\{x_2[n]\} = 3\,x_2[n] + 2\,x_2[n-1]$$

The response of the system to the linear combination signal $x[n] = \alpha_1\,x_1[n] + \alpha_2\,x_2[n]$ is computed as

$$\begin{aligned} y[n] &= \text{Sys}\{\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\} \\ &= 3\,(\alpha_1\,x_1[n] + \alpha_2\,x_2[n]) + 2\,(\alpha_1\,x_1[n-1] + \alpha_2\,x_2[n-1]) \\ &= \alpha_1\,(3\,x_1[n] + 2\,x_1[n-1]\,) + \alpha_2\,(3\,x_2[n] + 2\,x_2[n-1]\,) \\ &= \alpha_1\,y_1[n] + \alpha_2\,y_2[n] \end{aligned}$$

Superposition principle holds, and therefore the system in question is linear.

Example 3.1 (b) – Solution

b. Again using the test signals $x_1[n]$ and $x_2[n]$ we have

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = 3\,x_1[n] + 2\,x_1[n+1]\,x_1[n-1]$$

and

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = 3\,x_2[n] + 2\,x_2[n+1]\,x_2[n-1]$$

Use of the linear combination signal $x[n] = \alpha_1\,x_1[n] + \alpha_2\,x_2[n]$ as input to the system yields the output signal

$$\begin{aligned} y[n] =& \text{Sys}\left\{\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right\} \\ =& 3\,\left(\alpha_1\,x_1[n] + \alpha_2 x_2[n]\right) \\ &+ 2\,\left(\alpha_1\,x_1[n+1] + \alpha_2\,x_2[n+1]\right)\,\left(\alpha_1\,x_1[n-1] + \alpha_2\,x_2[n-1]\right) \end{aligned}$$

In this case the superposition principle does not hold true. The system in part (b) is therefore not linear.

Example 3.1 (c) – Solution

c. The responses of the system to the two test signals are

$$y_1[n] = \text{Sys}\{x_1[n]\} = a^{-n}\, x_1[n]$$

and

$$y_2[n] = \text{Sys}\{x_2[n]\} = a^{-n}\, x_2[n]$$

and the response to the linear combination signal $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ is

$$\begin{aligned} y[n] =& \text{Sys}\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\} \\ =& a^{-n}\, (\alpha_1\, x_1[n] + \alpha_2\, x_2[n]) \\ =& \alpha_1\, a^{-n}\, x_1[n] + \alpha_2\, a^{-n}\, x_2[n] \\ =& \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

Time Invariance

- Let a discrete-time system be described with the input-output relationship

$$y[n] = Sys\{x[n]\}$$

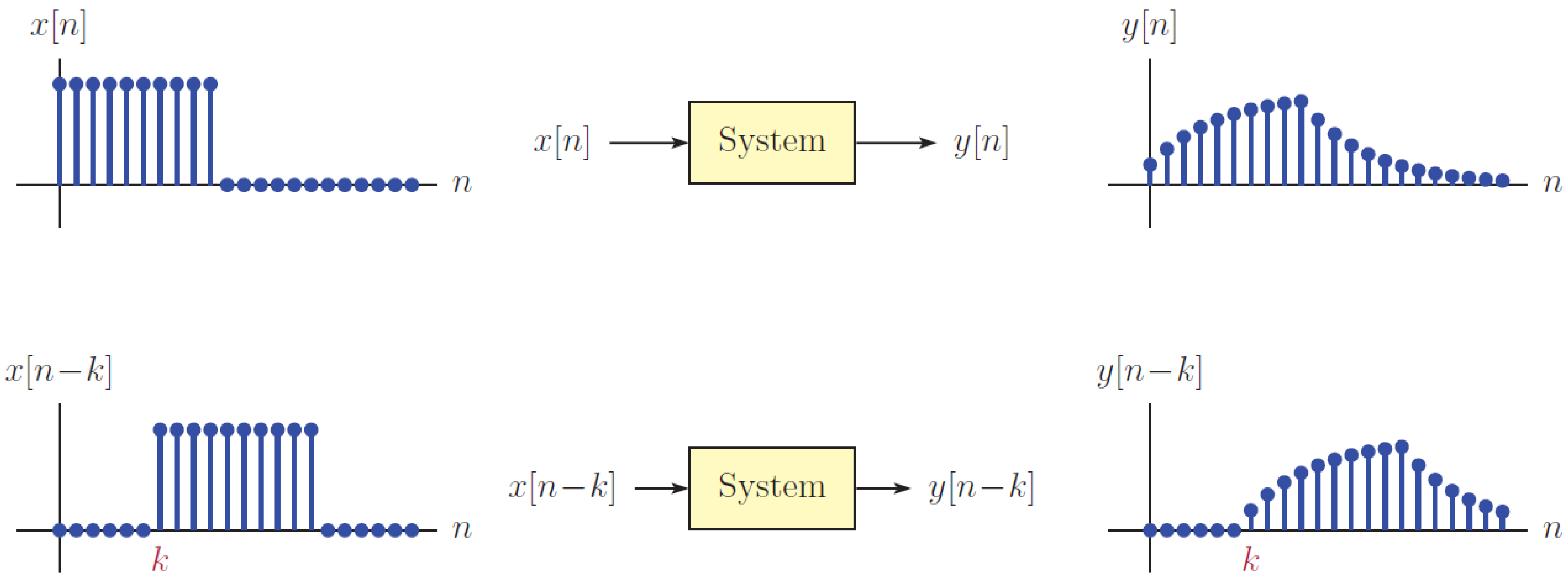
- For the system to be considered time-invariant, the only effect of time-shifting the input signal should be to cause an equal amount of time shift in the output signal.

$$Sys\{x[n]\} = y[n] \quad \text{implies that} \quad Sys\{x[n-k]\} = y[n-k]$$

Time Invariance

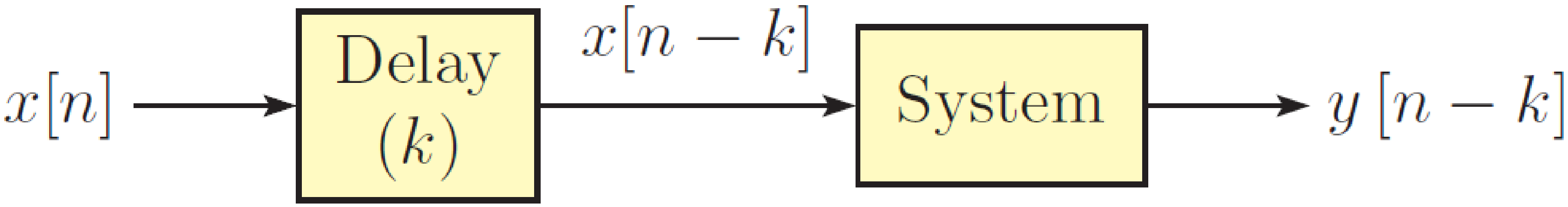
- Condition for time-invariance:

$$Sys\{x[n]\} = y[n] \quad \text{implies that} \quad Sys\{x[n-k]\} = y[n-k]$$

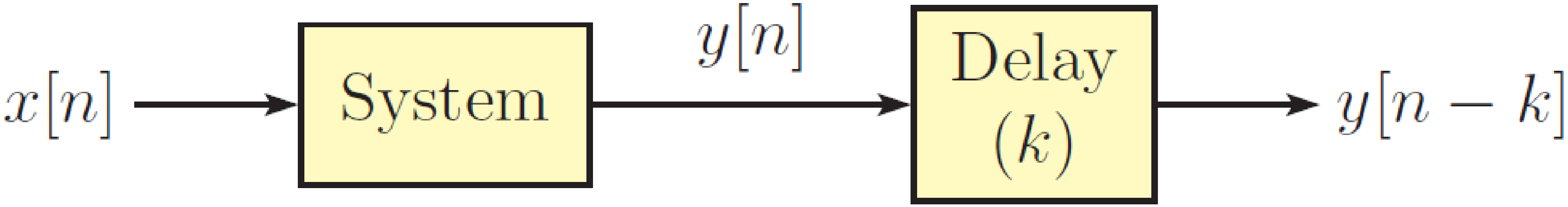


Time Invariance

- The time-invariant nature of a system can be characterized by the equivalence of the two configurations shown in Figure.



(a)



Example 3.2

For each of the discrete-time systems described below, determine whether the system is time-invariant or not:

a. $y[n] = y[n-1] + 3\,x[n]$

b. $y[n] = x[n]\,y[n-1]$

c. $y[n] = n\,x[n-1]$

Example 3.2 (a) – Solution

a. We will test the time-invariance property of the system by time-shifting both the input and the output signals by the same number of samples, and see if the input–output relationship still holds. Replacing the index n by $n-k$ in the arguments of all input and output terms we obtain

$$\text{Sys}\left\{x[n-k]\right\} = y[n-k-1] + 3\,x[n-k] = y[n-k]$$

The input–output relationship holds, therefore the system is time-invariant.

Example 3.2 (b) – Solution

b. Proceeding in a similar fashion we have

$$\text{Sys}\left\{x[n-k]\right\} = x[n-k]\,y[n-k-1] = y[n-k]$$

This system is time-invariant as well.

Example 3.2 (c) – Solution

c. Replacing the index n by $n-k$ in the arguments of all input and output terms yields

$$\text{Sys}\{x[n-k]\} = n\,x[n-k-1] \neq y[n-k]$$

This system is clearly not time-invariant since the input–output relationship no longer holds after input and output signals are time-shifted.
Should we have included the factor n in the time shifting operation when we wrote the response of the system to a time-shifted input signal? In other words, should we have written the response as

$$\text{Sys}\{x[n-k]\} \stackrel{?}{=} (n-k)\,x[n-k-1]$$

The answer is no. The factor n that multiplies the input signal is part of the system definition and not part of either the input or the output signal. Therefore we cannot include it in the process of time-shifting input and output signals.

Problem 3.1

A number of discrete-time systems are specified below in terms of their input-output relationships.

For each case determine if the system is linear and/or time-invariant.

a. $y[n] = x[n]\, u[n]$

c. $y[n] = 3\, x[n] + 5\, u[n]$

e. $y[n] = \cos(0.2\pi n)\, x[n]$

f. $y[n] = x[n] + 3\, x[n-1]$

Problem 3.1 (a) – Solution

a. $y[n] = x[n]\, u[n]$

a.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = x_1[n]\, u[n]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = x_2[n]\, u[n]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] &= \text{Sys}\left\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right\} \\ &= \left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right)\, u[n] \\ &= \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = x_1[n-m]\, u[n] \neq y_1[n-m]$$

The system is not time-invariant.

Problem 3.1 (c) – Solution

c. $y[n] = 3\,x[n] + 5\,u[n]$

c.

$$y_1[n] = \text{Sys}\left\{x_1[n]\right\} = 3\,x_1[n] + 5\,u[n]$$

$$y_2[n] = \text{Sys}\left\{x_2[n]\right\} = 3\,x_2[n] + 5\,u[n]$$

Using $x[n] = \alpha_1\,x_1[n] + \alpha_2\,x_2[n]$ as input we obtain

$$\begin{aligned} y[n] &= \text{Sys}\left\{\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right\} \\ &= 3\left(\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\right) + 5\,u[n] \\ &\neq \alpha_1\,y_1[n] + \alpha_2\,y_2[n] \end{aligned}$$

The system is not linear.

$$\text{Sys}\left\{x_1[n-m]\right\} = 3\,x_1[n-m] + 5\,u[n] \neq y_1[n-m]$$

The system is not time-invariant.

Problem 3.1 (e) – Solution

e. $y[n] = \cos(0.2\pi n)\, x[n]$

e.

$$y_1[n] = \text{Sys}\{x_1[n]\} = \cos(0.2\pi n)\; x_1[n]$$

$$y_2[n] = \text{Sys}\{x_2[n]\} = \cos(0.2\pi n)\; x_2[n]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] &= \text{Sys}\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\} \\ &= \cos(0.2\pi n)\, \left(\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\right) \\ &= \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\{x_1[n-m]\} = \cos(0.2\pi n)\; x_1[n-m] \neq y_1[n-m]$$

The system is not time-invariant.

Problem 3.1 (f) – Solution

f. $y[n] = x[n] + 3\,x[n-1]$

f.

$$y_1[n] = \text{Sys}\{x_1[n]\} = x_1[n] + 3\,x_1[n-1]$$

$$y_2[n] = \text{Sys}\{x_2[n]\} = x_2[n] + 3\,x_2[n-1]$$

Using $x[n] = \alpha_1\,x_1[n] + \alpha_2\,x_2[n]$ as input we obtain

$$\begin{aligned} y[n] &= \text{Sys}\{\alpha_1\,x_1[n] + \alpha_2\,x_2[n]\} \\ &= (\alpha_1\,x_1[n] + \alpha_2\,x_2[n]) + 3\,(\alpha_1\,x_1[n-1] + \alpha_2\,x_2[n-1]) \\ &= \alpha_1\,y_1[n] + \alpha_2\,y_2[n] \end{aligned}$$

The system is linear.

$$\text{Sys}\{x_1[n-m]\} = x_1[n-m] + 3\,x_1[n-m-1] = y_1[n-m]$$

The system is time-invariant.

Problem 3.2 (b)

Determine if the system is linear or not.

b. $$y[n] = \sum_{k=0}^{n} x[k]$$

Problem 3.2 (b) – Solution

b.

$$y_1[n] = \text{Sys}\{x_1[n]\} = \sum_{k=0}^{n} x_1[k]$$

$$y_2[n] = \text{Sys}\{x_2[n]\} = \sum_{k=0}^{n} x_2[k]$$

Using $x[n] = \alpha_1\, x_1[n] + \alpha_2\, x_2[n]$ as input we obtain

$$\begin{aligned} y[n] &= \text{Sys}\{\alpha_1\, x_1[n] + \alpha_2\, x_2[n]\} \\ &= \sum_{k=0}^{n} \left(\alpha_1\, x_1[k] + \alpha_2\, x_2[k]\right) \\ &= \alpha_1 \sum_{k=0}^{n} x_1[k] + \alpha_2 \sum_{k=0}^{n} x_2[k] \\ &= \alpha_1\, y_1[n] + \alpha_2\, y_2[n] \end{aligned}$$

The system is linear.

Problem 3.3

3.3. Consider the cascade combination of two systems shown in Fig. P.3.3(a).

Figure P. 3.3

a. Let the input-output relationships of the two subsystems be given as

$$\text{Sys}_1\{x[n]\} = 3\,x[n] \quad \text{and} \quad \text{Sys}_2\{w[n]\} = w[n-2]$$

Write the relationship between $x[n]$ and $y[n]$.

b. Let the order of the two subsystems be changed as shown in Fig. P.3.3(b). Write the relationship between $x[n]$ and $\bar{y}[n]$. Does changing the order of two subsystems change the overall input-output relationship of the system?

Problem 3.3 – Solution

a.

$$w[n] = 3\,x[n]$$

$$y[n] = w[n-2] = 3\,x[n-2]$$

b.

$$\bar{w}[n] = x[n-2]$$

$$\bar{y}[n] = 3\,\bar{w}[n] = 3\,x[n-2]$$

Input-output relationship of the system does not change when the order of the two subsystems is changed.

DTLTI Systems

- Discrete-time systems that are both linear and time-invariant will play an important role in the rest of this textbook.
- We will develop time- and frequency-domain analysis and design techniques for working with such systems.
- To simplify the terminology, we will use the acronym DTLTI to refer to discrete-time linear and time-invariant systems.

Difference Equations for Discrete-Time Systems

- In chapter 2, we have discussed methods of representing continuous-time systems with differential equations.
- Using a similar approach, discrete-time systems can be modeled with difference equations involving current, past, or future samples of input and output signals.
- We will focus on difference equations for DTLTI systems.

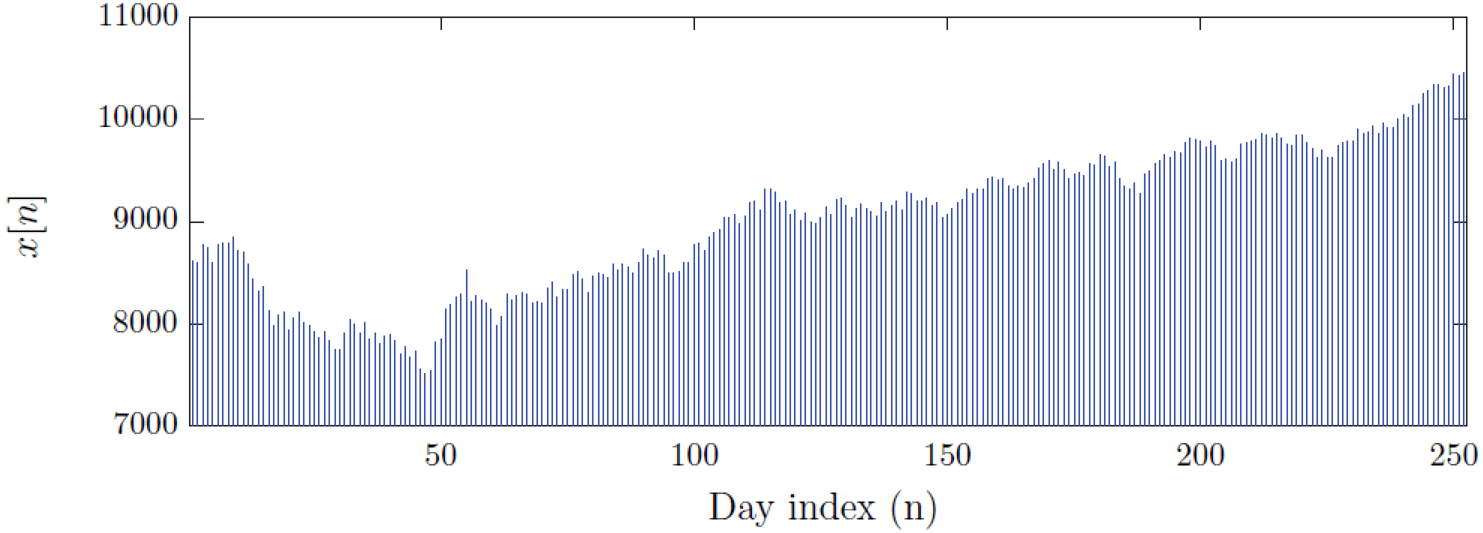
Moving-Average Filter

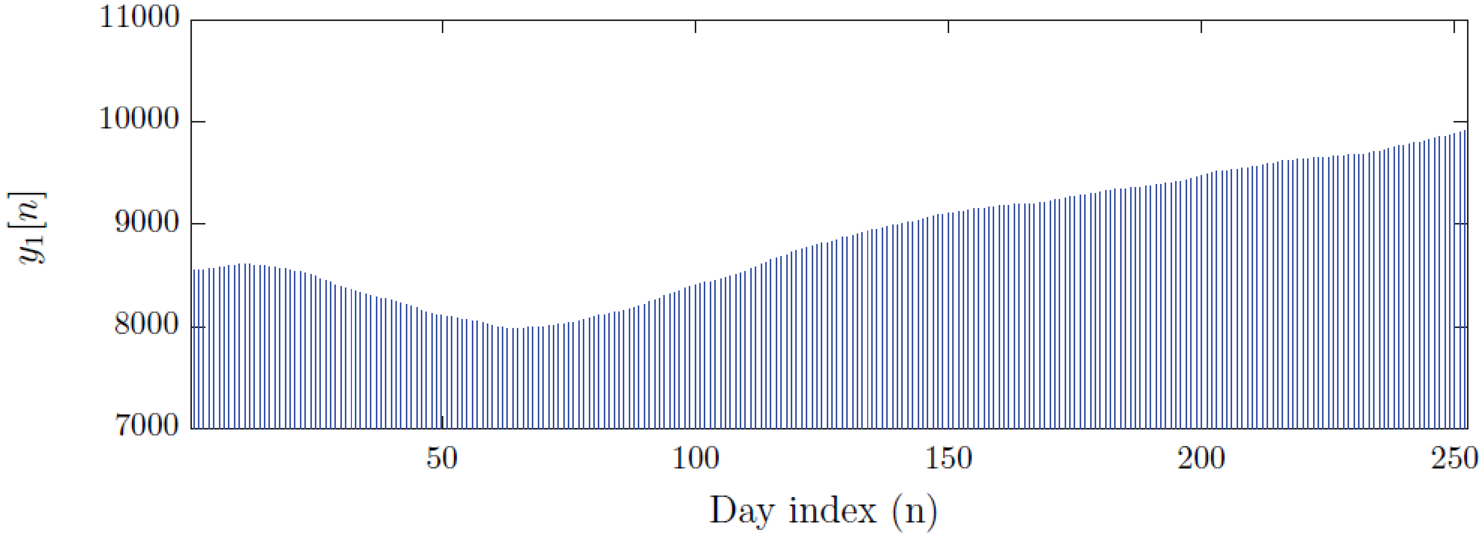
- A length-N moving average filter is a simple system that produces an output equal to the arithmetic average of the most recent N samples of the input signal.
- The general expression for the length-N moving average filter is

$$y[n] = \frac{x[n] + x[n-1] + \ldots + x[n-(N-1)]}{N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$$

Moving-Average Filter





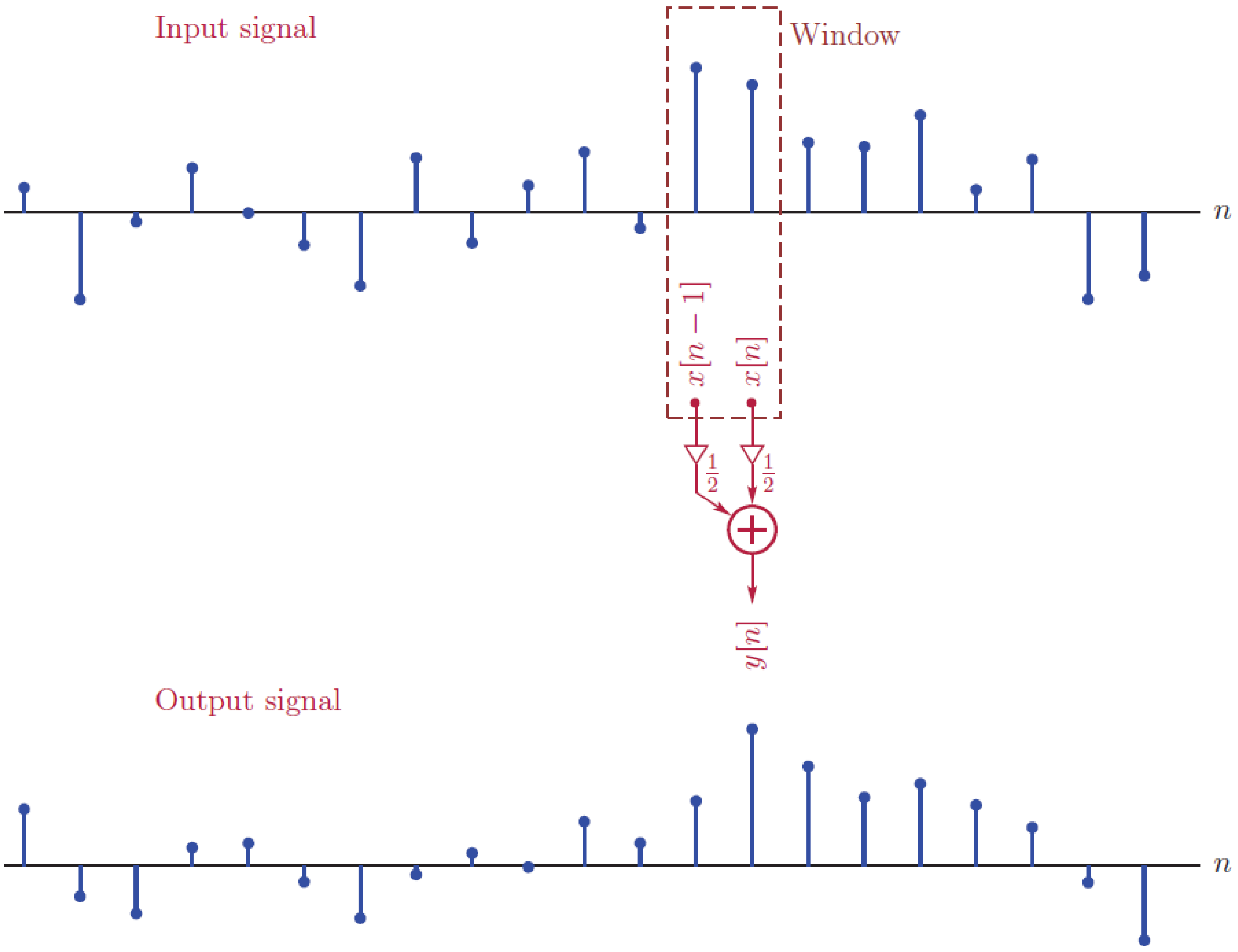
Example 3.4: Length-2 Moving-Average Filter

- A length-2 moving average filter produces an output by averaging the current input sample and the previous input sample.
- This action translates to a difference equation in the form

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

$$y[n] = \frac{1}{2} x[n] + \frac{1}{2} x[n-1]$$

Example 3.4: Length-2 Moving-Average Filter



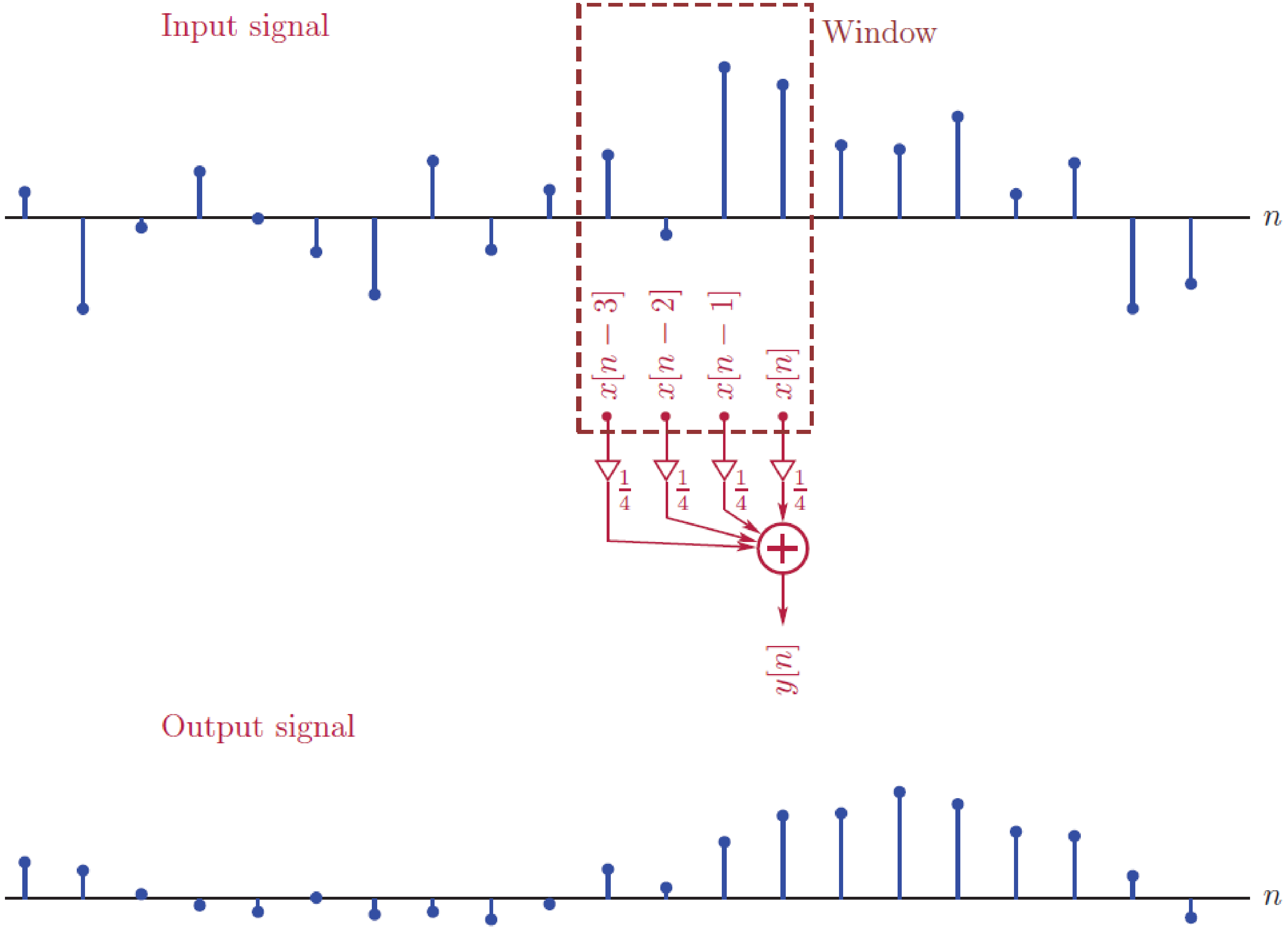
Example 3.4: Length-4 Moving-Average Filter

- A length-4 moving average filter produces an output by averaging the current input sample and the previous three input samples.
- This action translates to a difference equation in the form

$$y[n] = \frac{x[n] + x[n-1] + x[n-2] + x[n-3]}{4}$$

$$y[n] = \frac{1}{4}x[n] + \frac{1}{4}x[n-1] + \frac{1}{4}x[n-2] + \frac{1}{4}x[n-3]$$

Example 3.4: Length-4 Moving-Average Filter



Problem 3.7

3.7. The discrete-time signal

$$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{1.7}, 2.3, 3.1, 3.3, 3.7, 2.9, 2.2, 1.4, 0.6, -0.2, 0.4 \}$$

is used as input to a length-2 moving average filter. Determine the response $y[n]$ for $n = 0, \ldots, 9$. Use $x[-1] = 0$.

Problem 3.7 – Solution

$$y[0] = \frac{x[0] + x[-1]}{2} = \frac{1.7 + 0}{2} = 0.85$$

$$y[1] = \frac{x[1] + x[0]}{2} = \frac{2.3 + 1.7}{2} = 2$$

$$y[2] = \frac{x[2] + x[1]}{2} = \frac{3.1 + 2.3}{2} = 2.7$$

$$y[3] = \frac{x[3] + x[2]}{2} = \frac{3.3 + 3.1}{2} = 3.2$$

$$y[4] = \frac{x[4] + x[3]}{2} = \frac{3.7 + 3.3}{2} = 3.5$$

$$y[5] = \frac{x[5] + x[4]}{2} = \frac{2.9 + 3.7}{2} = 3.3$$

$$y[6] = \frac{x[6] + x[5]}{2} = \frac{2.2 + 2.9}{2} = 2.55$$

$$y[7] = \frac{x[7] + x[6]}{2} = \frac{1.4 + 2.2}{2} = 1.8$$

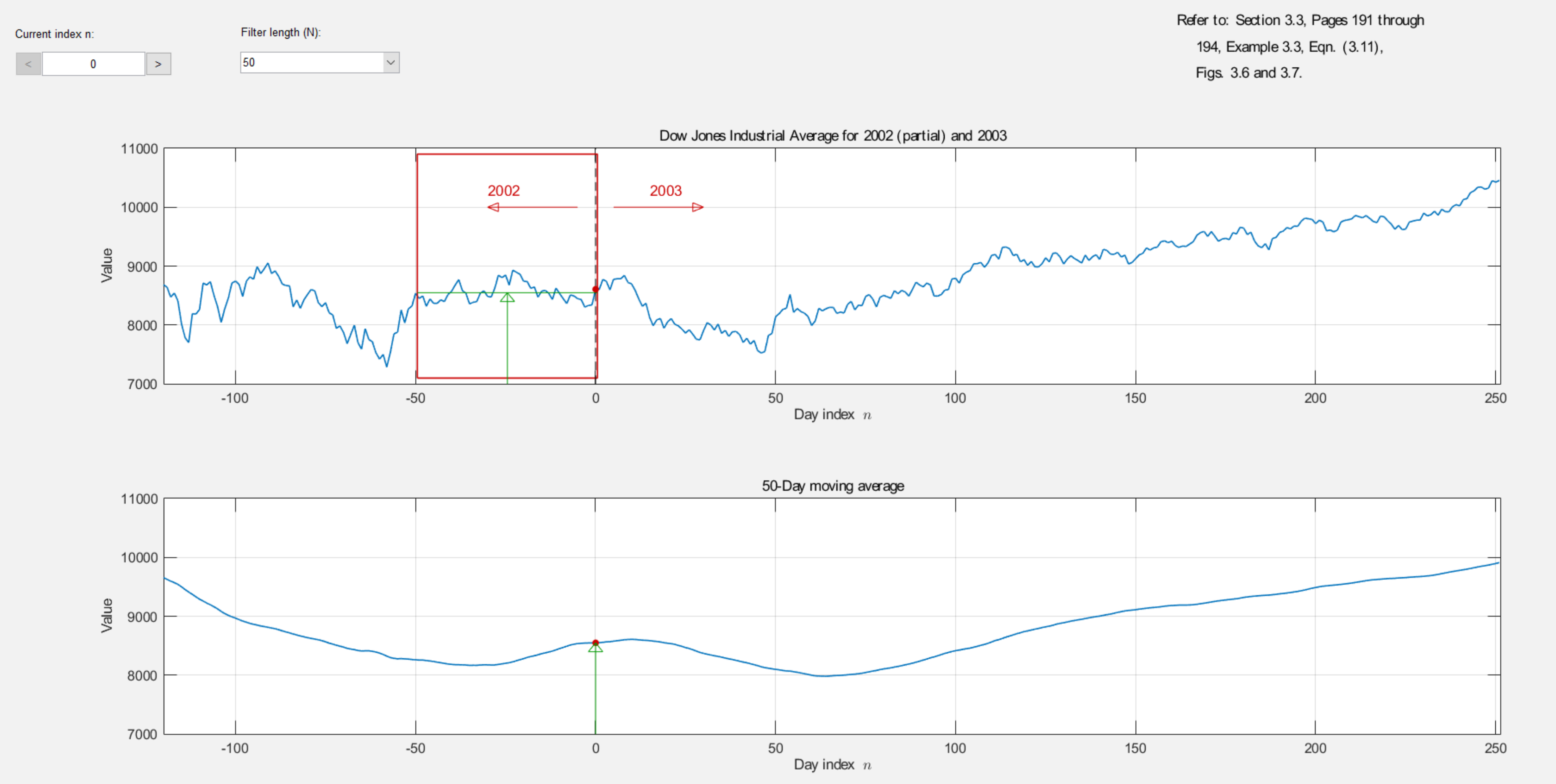
$$y[8] = \frac{x[8] + x[7]}{2} = \frac{0.6 + 1.4}{2} = 1$$

$$y[9] = \frac{x[9] + x[8]}{2} = \frac{-0.2 + 0.6}{2} = 0.2$$

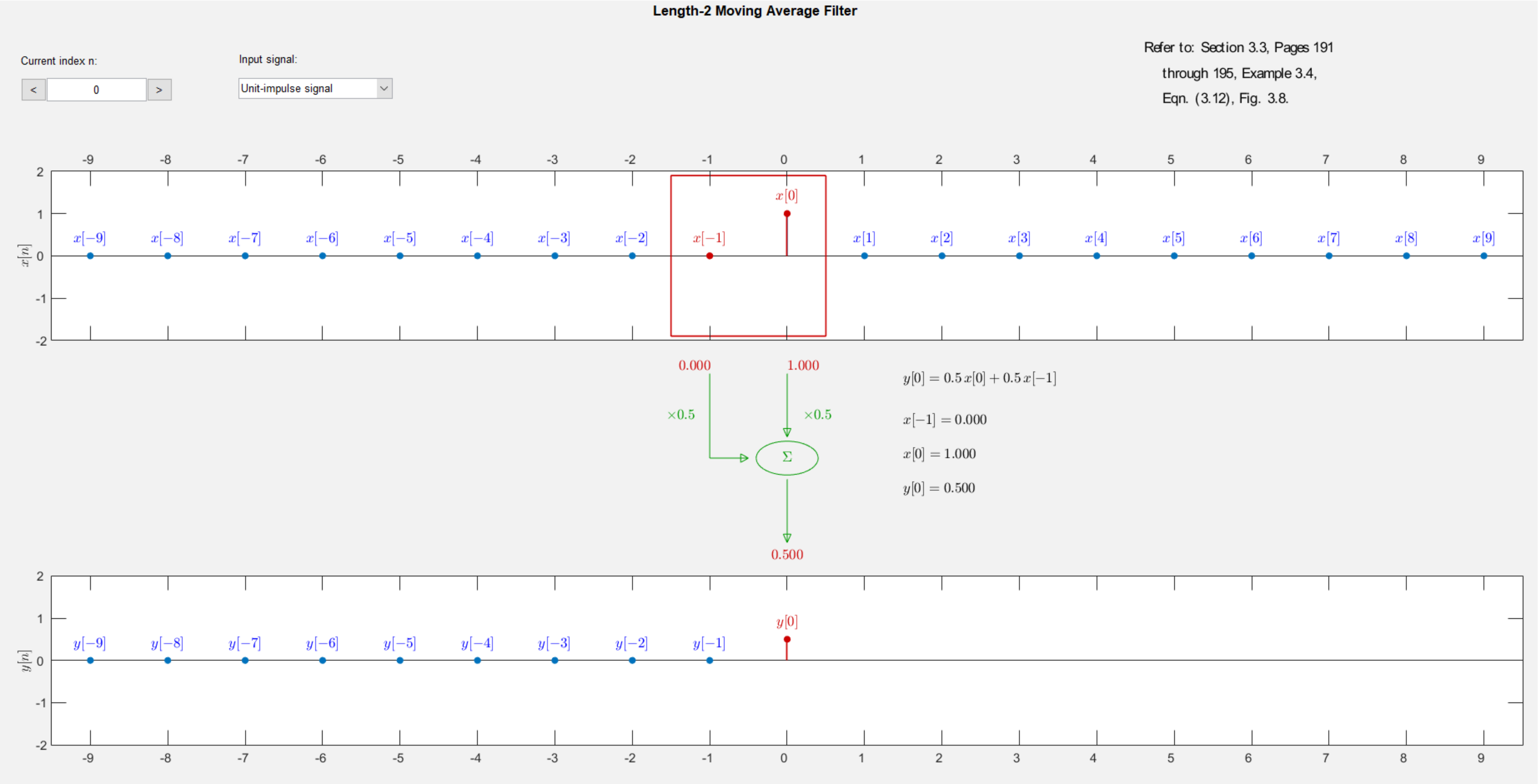
$$y[10] = \frac{x[10] + x[9]}{2} = \frac{0.4 - 0.2}{2} = 0.1$$

$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{0.85}, 2, 2.7, 3.2, 3.5, 3.3, 2.55, 1.8, 1, 0.2, 0.1\}$$

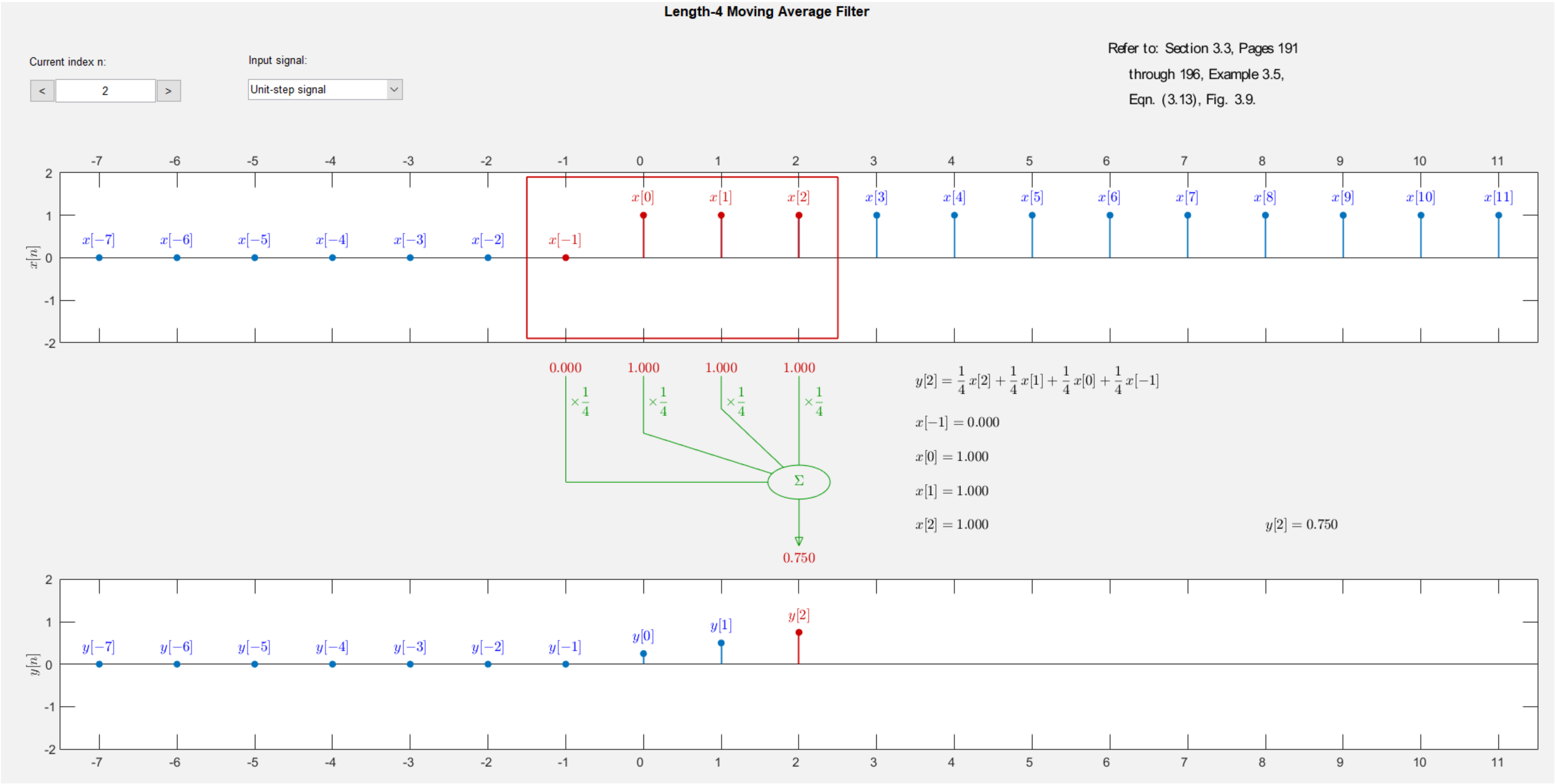
Interactive Demo: ma_demo1



Interactive Demo: ma_demo2



Interactive Demo: ma_demo3



Continuous-Time Convolution

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.

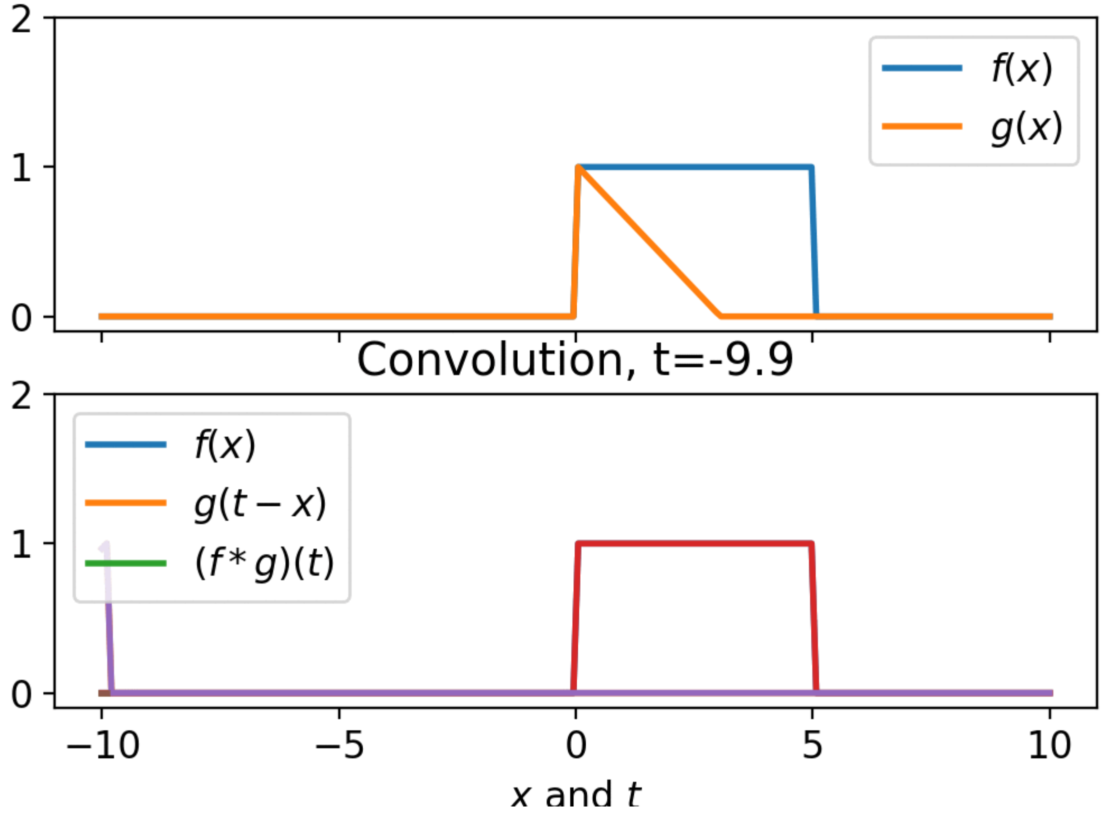


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Image Processing: Convolution vs. Correlation

- Correlation consists of moving the center of a kernel over an image, and computing the sum of products at each location.
- Convolution is the same as correlation, except that the correlation kernel is rotated by 180°.

Property	Convolution	Correlation
Commutative	$f \star g = g \star f$	—
Associative	$f \star (g \star h) = (f \star g) \star h$	—
Distributive	$f \star (g + h) = (f \star g) + (f \star h)$	$f \,☆\, (g + h) = (f \,☆\, g) + (f \,☆\, h)$

Image Processing: Smoothing (Lowpass) Spatial Filters

- Smoothing (averaging) spatial filters are used to reduce sharp transitions in intensity.
- An obvious application of smoothing is noise reduction.
- Smoothing is used to reduce irrelevant detail in an image.

$\frac{1}{9} \times$

1	1	1
1	1	1
1	1	1

Image Processing: Smoothing (Lowpass) Spatial Filters

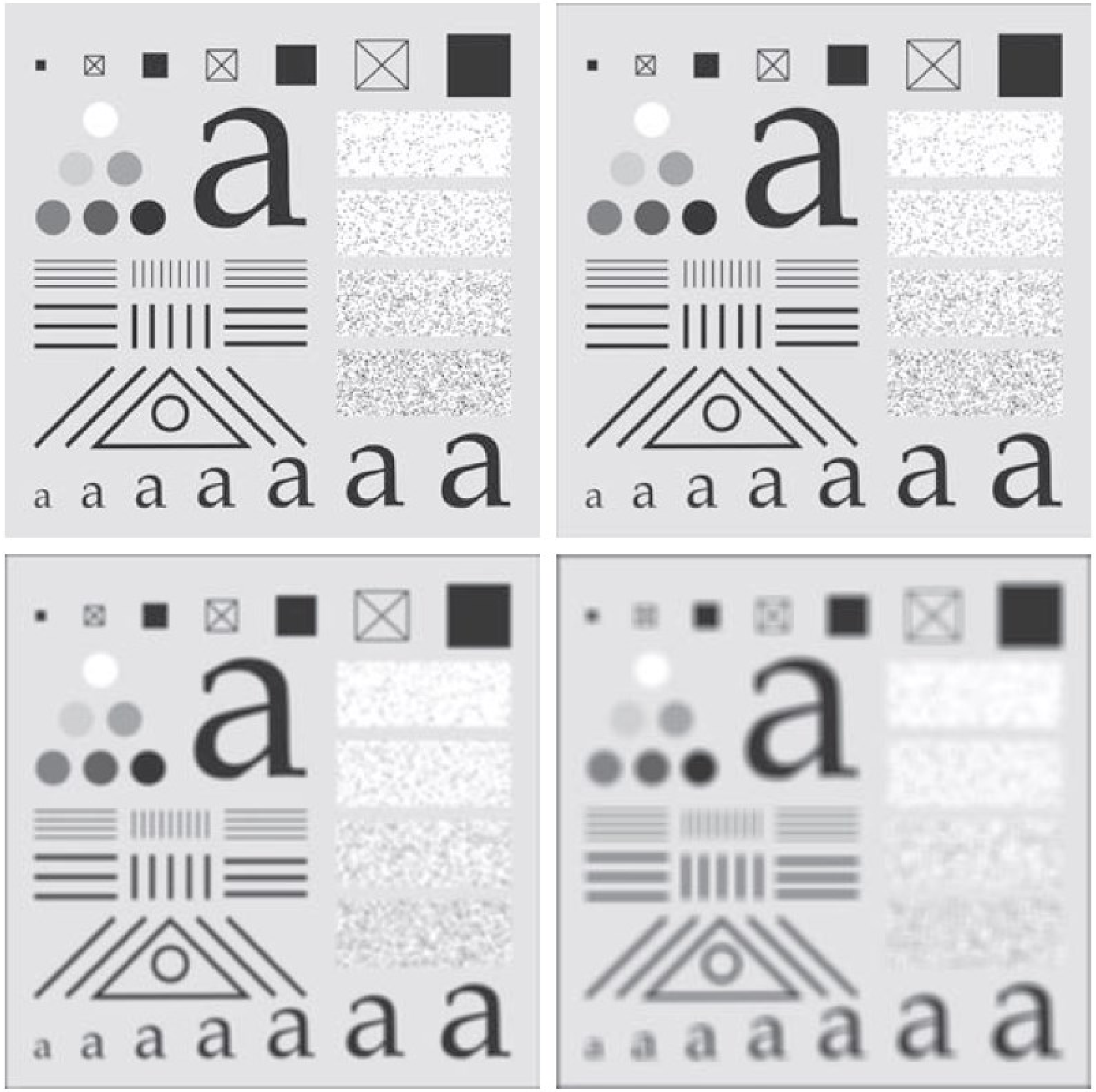


Image Processing: Sharpening (Highpass) Spatial Filters

- The simplest derivative operator (kernel) is the Laplacian, which, for a function (image) $f(x, y)$ of two variables, is defined as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

- In the x-direction, we have

$$\frac{\partial^2 f}{\partial x^2} = f(x+1, y) + f(x-1, y) - 2f(x, y)$$

- In the y-direction, we have

$$\frac{\partial^2 f}{\partial y^2} = f(x, y+1) + f(x, y-1) - 2f(x, y)$$

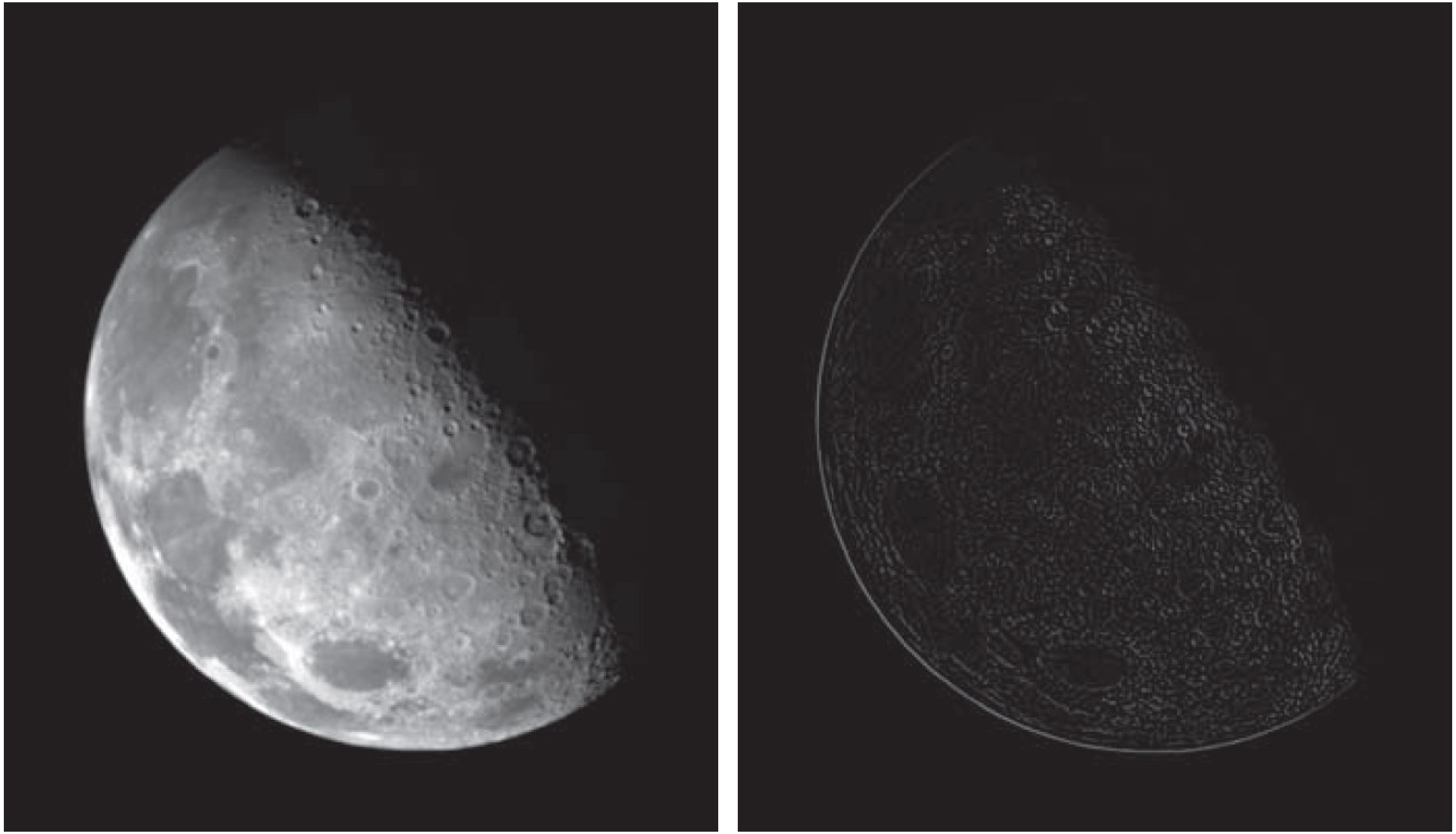
Image Processing: Sharpening (Highpass) Spatial Filters

- It follows from the preceding three equations that the discrete Laplacian of two variables is

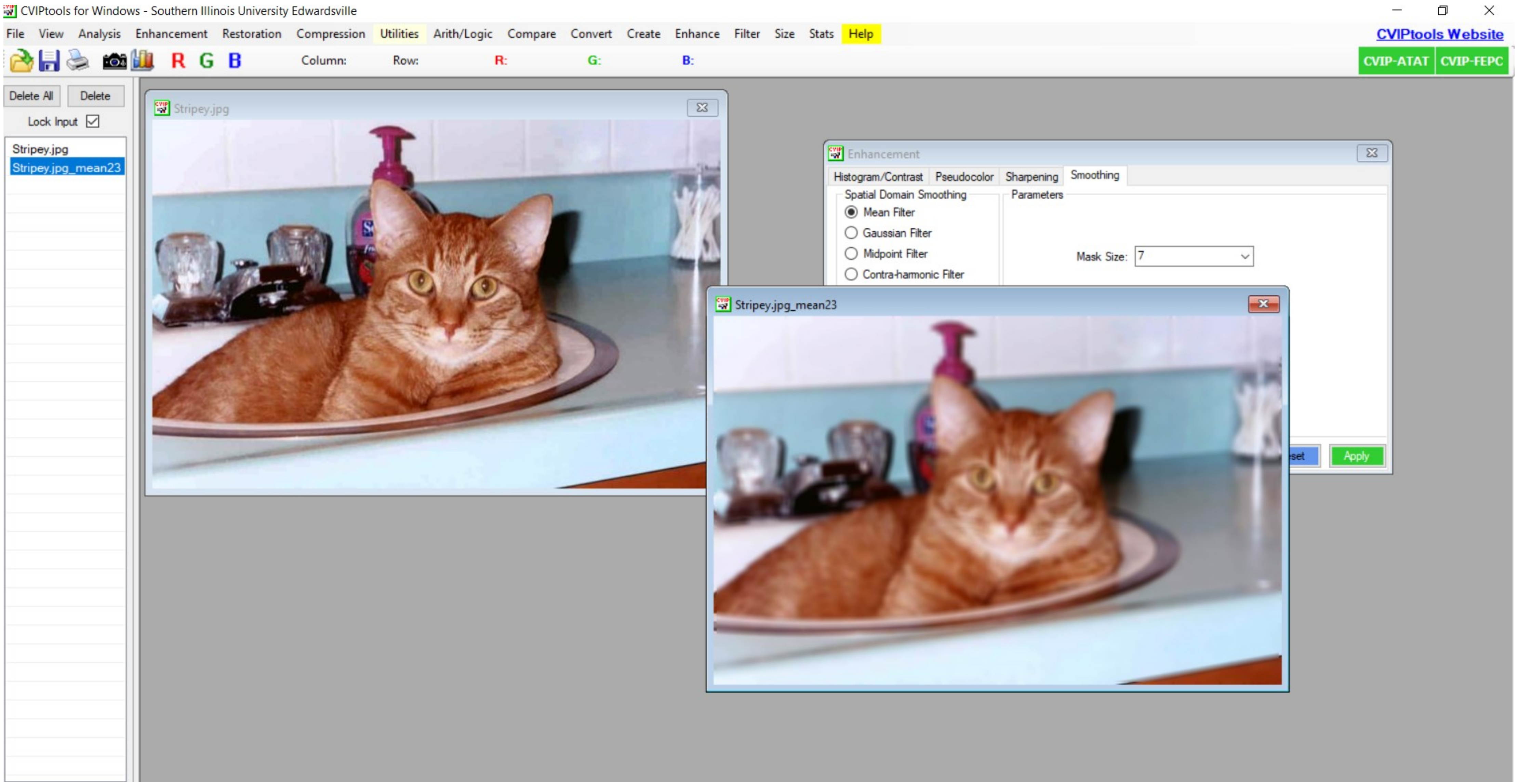
$$\nabla^2 f(x,y) = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)$$

0	1	0
1	−4	1
0	1	0

Image Processing: Sharpening (Highpass) Spatial Filters



CVIPtools



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File View Analysis Enhancement Restoration Compression Utilities Arith/Logic Compare Convert Create Enhance Filter Size Stats Help
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Mean Filter
Gaussian Filter
Midpoint Filter
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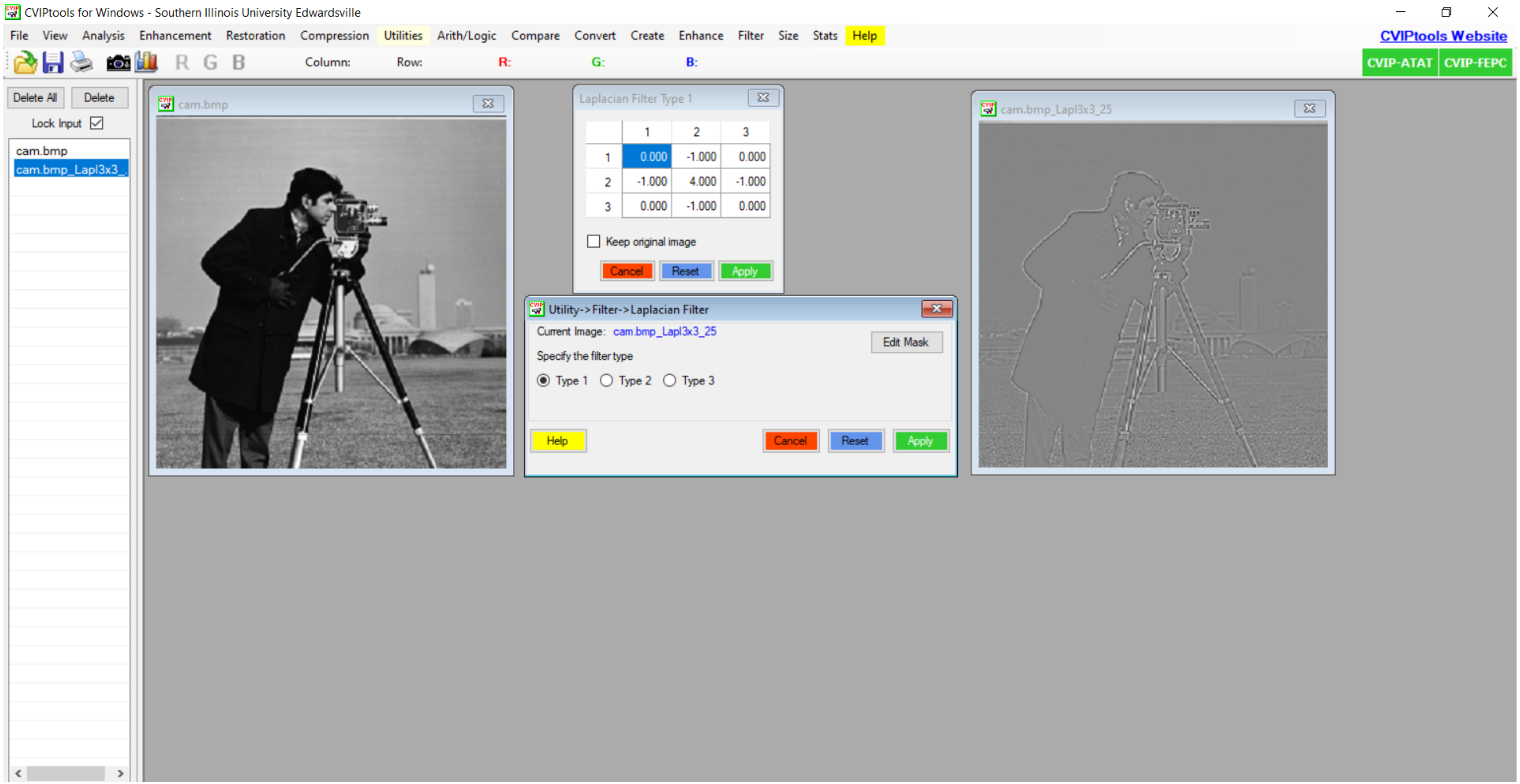
CVIPtools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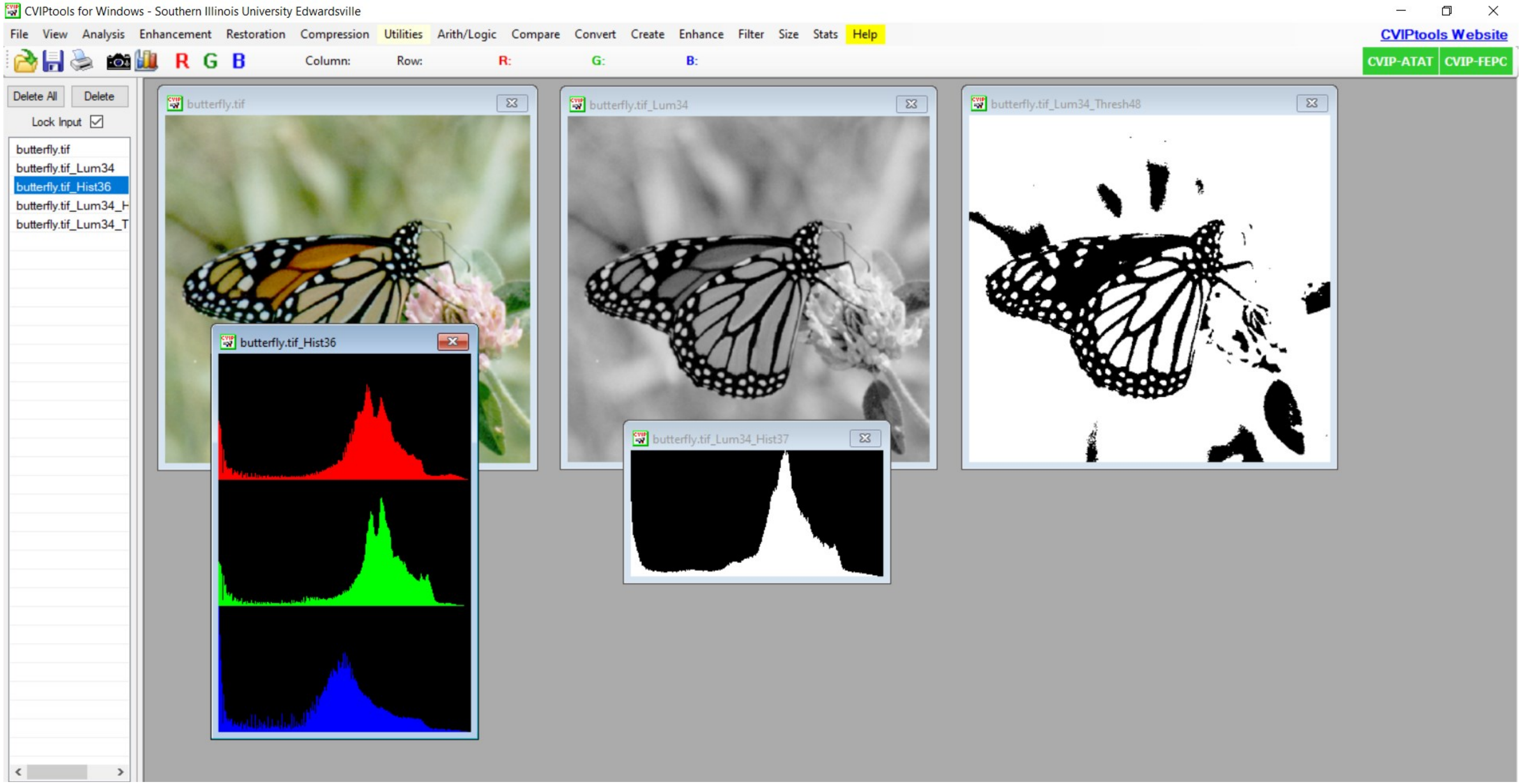
1 0.000 -1.000 0.000
2 -1.000 4.000 -1.000
3 0.000 -1.000 0.000
Keep original image
Cancel Reset Apply
cam.bmp_Lapl3x3_25
Utility->Filter->Laplacian Filter
Current Image: cam.bmp_Lapl3x3_25
Edit Mask
Specify the filter type
Type 1 Type 2 Type 3
Help Cancel Reset Apply

CVIPtools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butterfly.tif_Hist36
butterfly.tif_Lum34_H
butterfly.tif_Lum34_T
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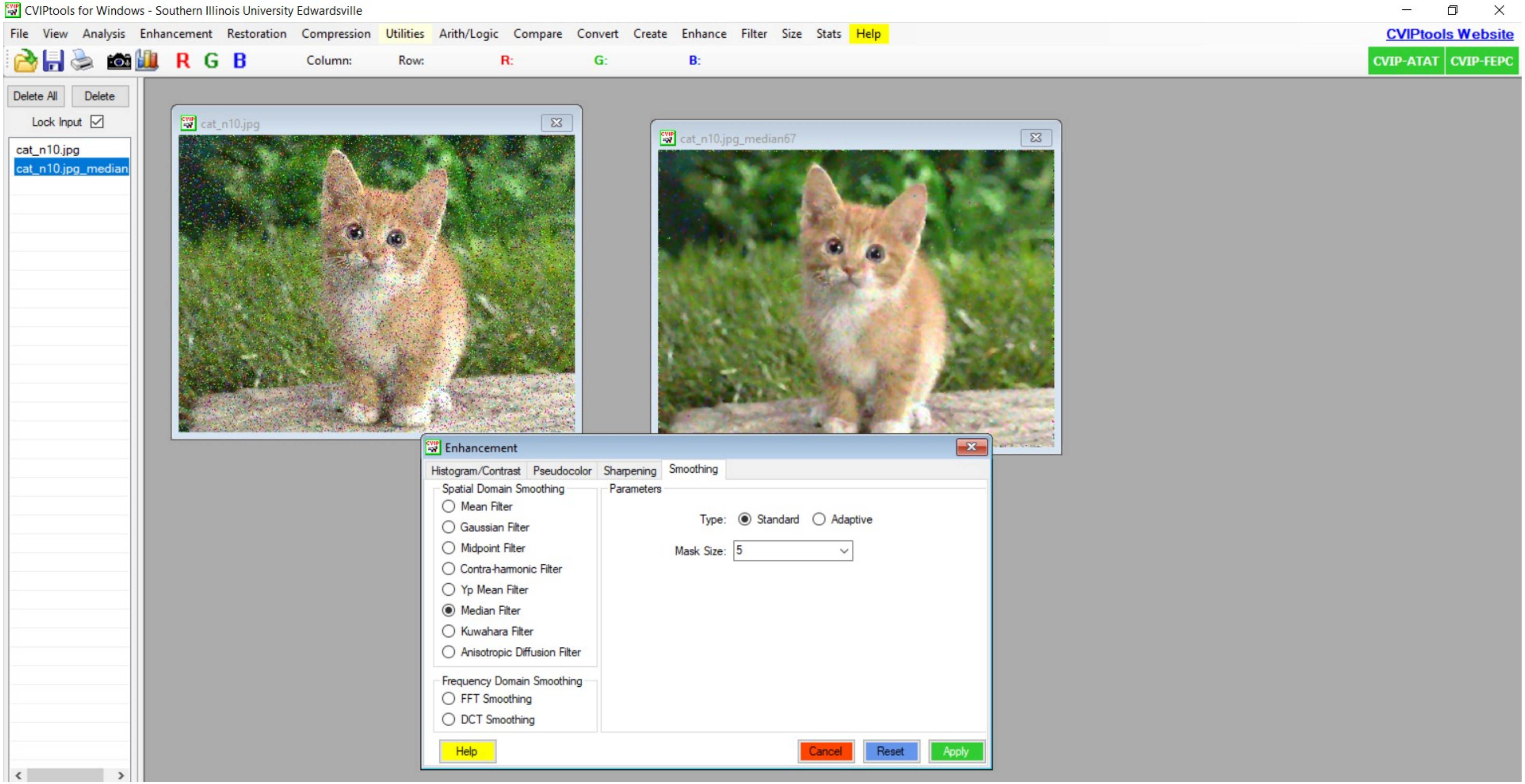
CVIPtools



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File View Analysis Enhancement Restoration Compression Utilities Arith/Logic Compare Convert Create Enhance Filter Size Stats Help
CVIPtools Website
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CVIP-ATAT CVIP-FEPC
Delete All Delete
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cat_n10.jpg_median67
Enhancement
Histogram/Contrast Pseudocolor Sharpening Smoothing
Spatial Domain Smoothing
Mean Filter
Gaussian Filter
Midpoint Filter
Contra-harmonic Filter
Yp Mean Filter
Median Filter
Kuwahara Filter
Anisotropic Diffusion Filter
Frequency Domain Smoothing
FFT Smoothing
DCT Smoothing
Parameters
Type: Standard Adaptive
Mask Size: 5
Help
Cancel
Reset
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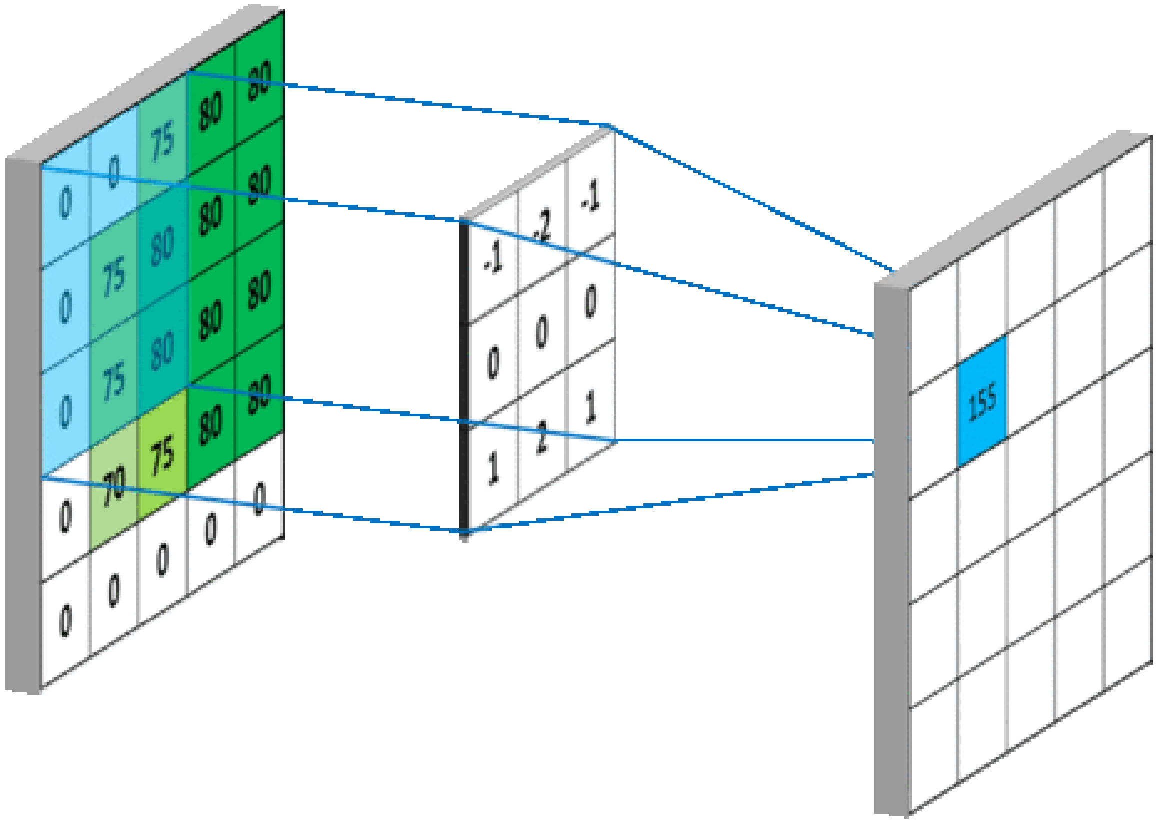
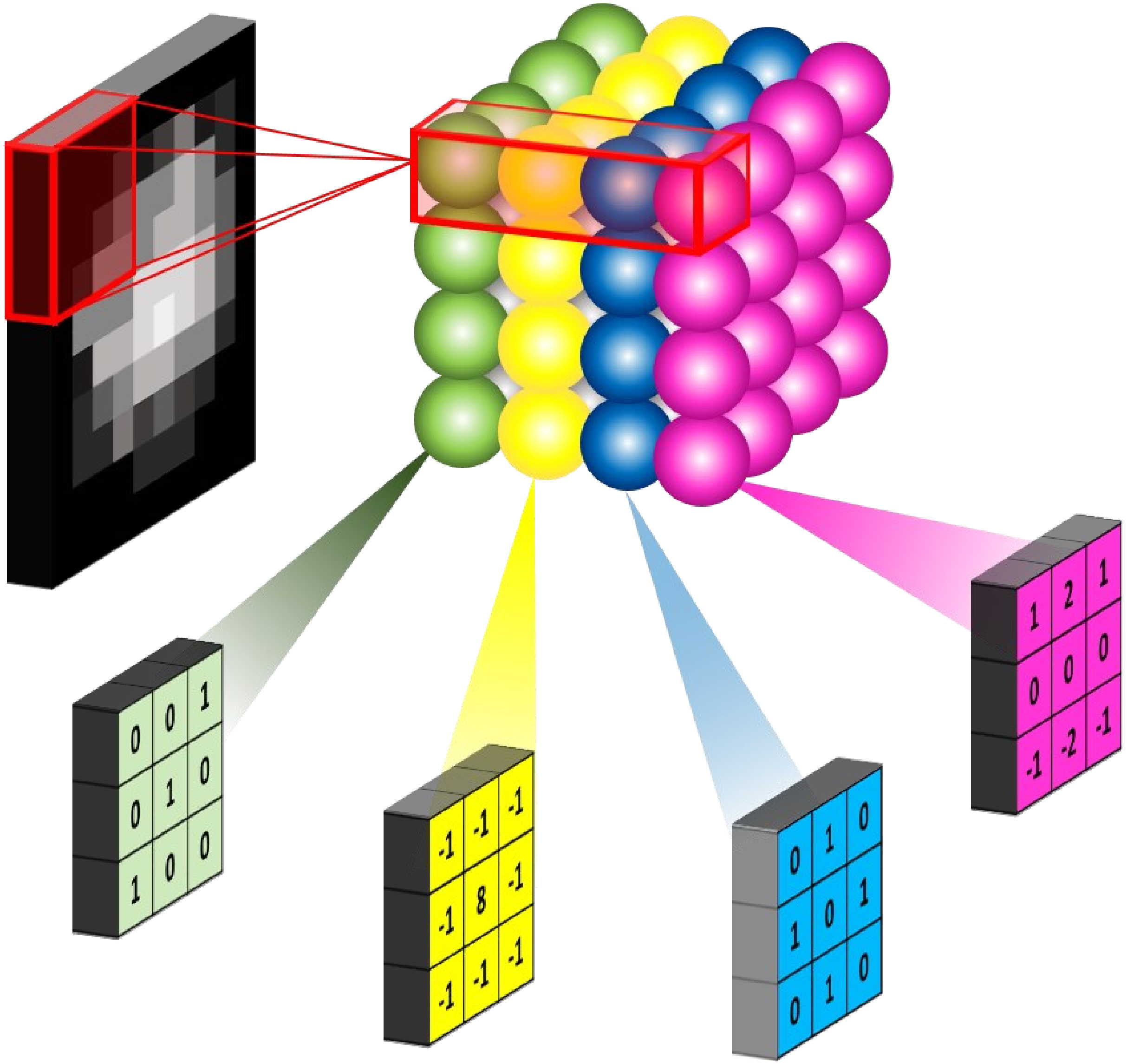
Convolutional Neural Networks

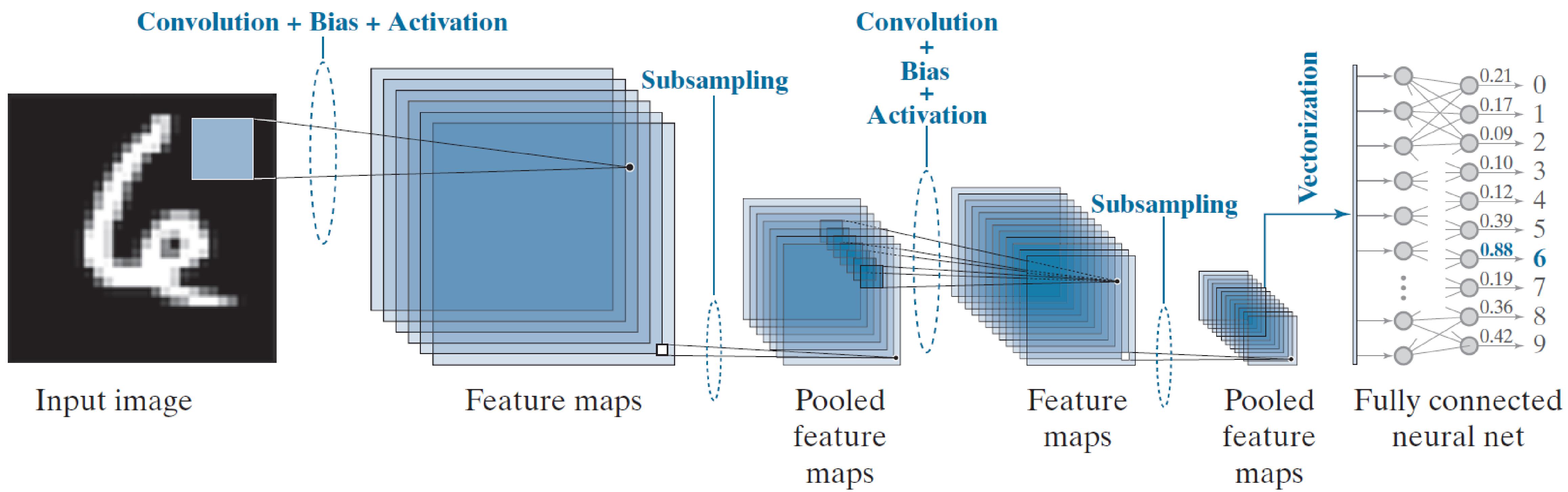


https://mlnotebook.github.io/post/CNN1/

Convolutional Neural Networks



Convolutional Neural Networks



Continuous-Time Convolution

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Discrete-Time Convolution

- The output signal $y[n]$ of a DTLTI system is obtained by convolving the input signal $x[n]$ and the impulse response $h[n]$ of the system.
- This relationship is expressed in compact notation as

$$y[n] = x[n] * h[n]$$

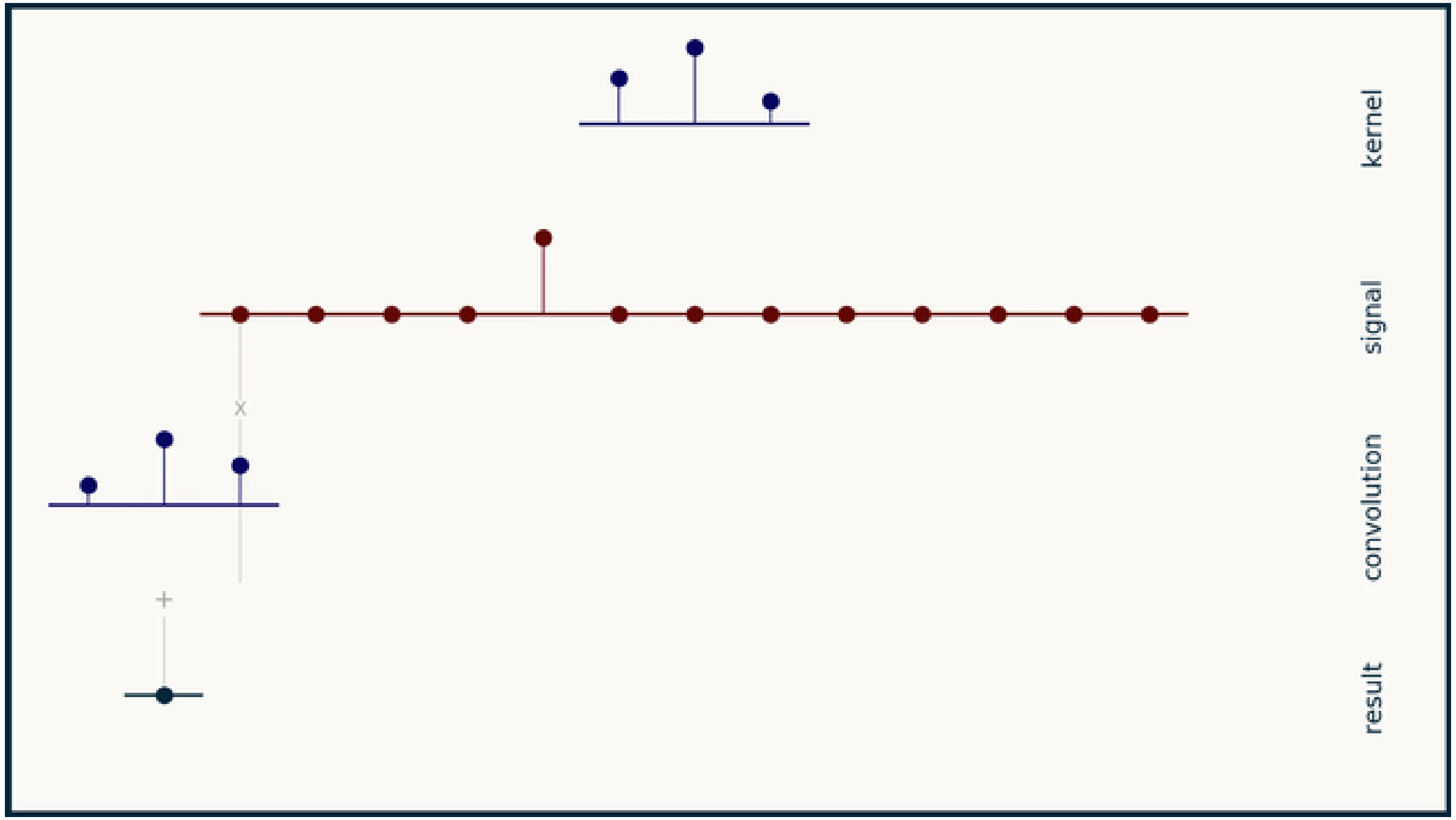
where the symbol $*$ represents the convolution operator.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Discrete-Time Convolution

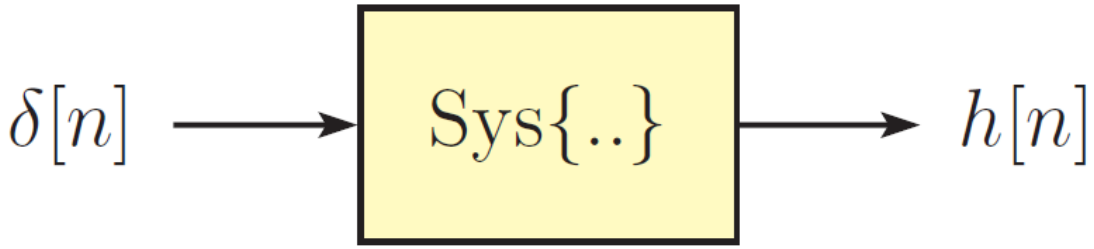
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



https://e2eml.school/convolution_one_d.html

Impulse Response

- A constant-coefficient linear difference equation is sufficient for describing a DTLTI system.
- The impulse response also constitutes a complete description of a DTLTI system.
- The response of a DTLTI system to any arbitrary input signal $x[n]$ can be uniquely determined from the knowledge of its impulse response.



Example 3.19

Example 3.19: A simple discrete-time convolution example

A discrete-time system is described through the impulse response

$$h[n] = \{ \underset{\substack{\uparrow \\ n=0}}{4}, 3, 2, 1 \}$$

Use the convolution operation to find the response of the system to the input signal

$$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{-3}, 7, 4 \}$$

Example 3.19 – Solution

Solution: Consider the convolution sum given by Eqn. (3.128). Let us express the terms inside the convolution summation, namely $x[k]$ and $h[n-k]$, as functions of k.

$$x[k] = \{ \underset{\underset{k=0}{\uparrow}}{-3}, 7, 4 \}$$

$$h[-k] = \{ 1, 2, 3, \underset{\underset{k=0}{\uparrow}}{4} \}$$

$$h[n-k] = \{ 1, 2, 3, \underset{\underset{k=n}{\uparrow}}{4} \}$$

In its general form both limits of the summation in Eqn. (3.128) are infinite. On the other hand, $x[k] = 0$ for negative values of the summation index k, so setting the lower limit of the summation to $k = 0$ would have no effect on the result. Similarly, the last significant sample of $x[k]$ is at index $k = 2$, so the upper limit can be changed to $k = 2$ without affecting the result as well, leading to

$$y[n] = \sum_{k=0}^{2} x[k]\, h[n-k] \qquad (3.131)$$

Example 3.19 – Solution

For $n = 0$:

$$y[0] = \sum_{k=0}^{0} x[k]\, h[0-k]$$

$$= x[0]\, h[0] = (-3)\ (4) = -12$$

For $n = 1$:

$$y[1] = \sum_{k=0}^{1} x[k]\, h[1-k]$$

$$= x[0]\, h[1] + x[1]\, h[0]$$

$$= (-3)\ (3) + (7)\ (4) = 19$$

Example 3.19 – Solution

For $n = 2$:

$$y[2] = \sum_{k=0}^{2} x[k]\, h[2-k]$$

$$= x[0]\, h[2] + x[1]\, h[1] + x[2]\, h[0]$$

$$= (-3)\ (2) + (7)\ (3) + (4)\ (4) = 31$$

For $n = 3$:

$$y[3] = \sum_{k=0}^{2} x[k]\, h[3-k]$$

$$= x[0]\, h[3] + x[1]\, h[2] + x[2]\, h[1]$$

$$= (-3)\ (1) + (7)\ (2) + (4)\ (3) = 23$$

Example 3.19 – Solution

For $n = 4$:

$$y[4] = \sum_{k=1}^{2} x[k]\, h[4-k]$$

$$= x[1]\, h[3] + x[2]\, h[2]$$

$$= (7)\ (1) + (4)\ (2) = 15$$

For $n = 5$:

$$y[5] = \sum_{k=2}^{2} x[k]\, h[5-k]$$

$$= x[2]\, h[3] = (4)\ (1) = 4$$

Thus the convolution result is

$$y[n] = \{ \underset{\substack{\uparrow \\ n=0}}{-12}\ ,\ 19,\ 31,\ 23,\ 15,\ 4 \}$$

Convolution Using MATLAB

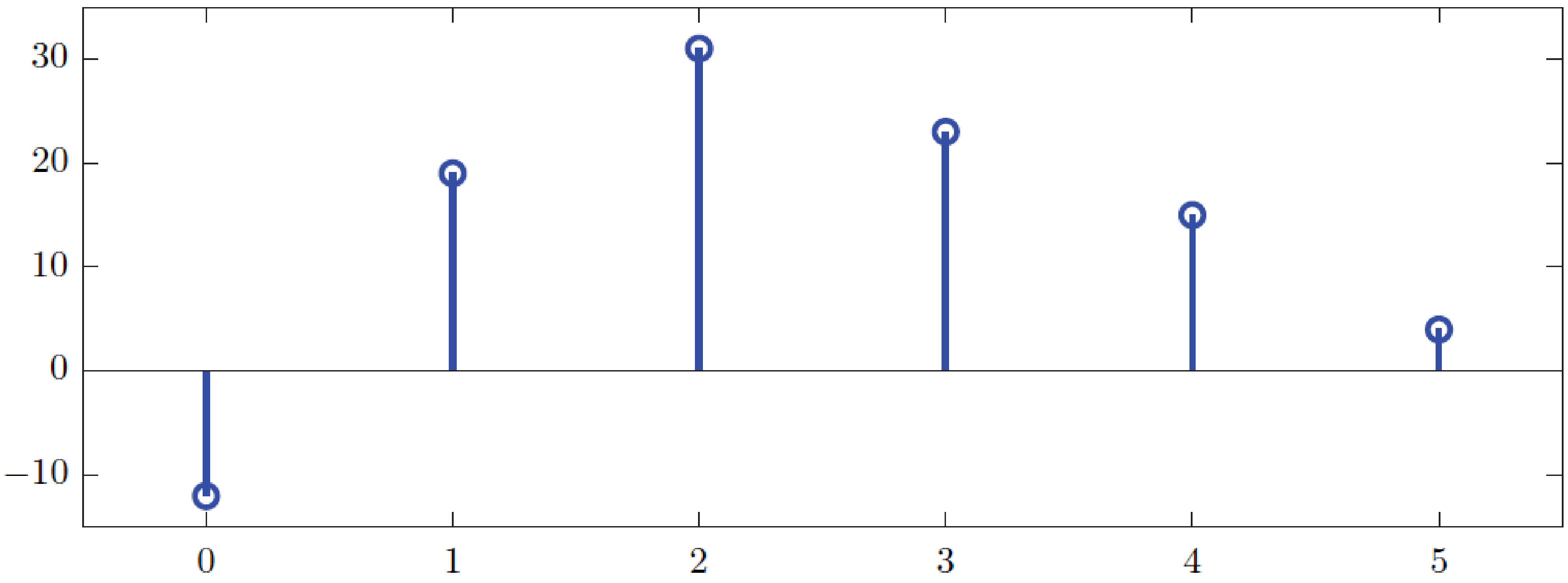
```
>> x = [-3, 7, 4];
>> h = [4, 3, 2, 1];

>> y = conv(x, h)
y =
    -12 19 31 23 15 4

>> y = conv(h, x)
y =
    -12 19 31 23 15 4
```

Convolution Using MATLAB

```
>> n = [0:5];
>> stem(n, y);
```



Problem 3.5

3.5. The response of a linear and time-invariant system to the input signal $x[n] = \delta[n]$ is given by

$$\text{Sys}\{\delta[n]\} = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 1, -1 \}$$

Determine the response of the system to the following input signals:

a. $x[n] = \delta[n] + \delta[n-1]$

b. $x[n] = \delta[n] - 2\,\delta[n-1] + \delta[n-2]$

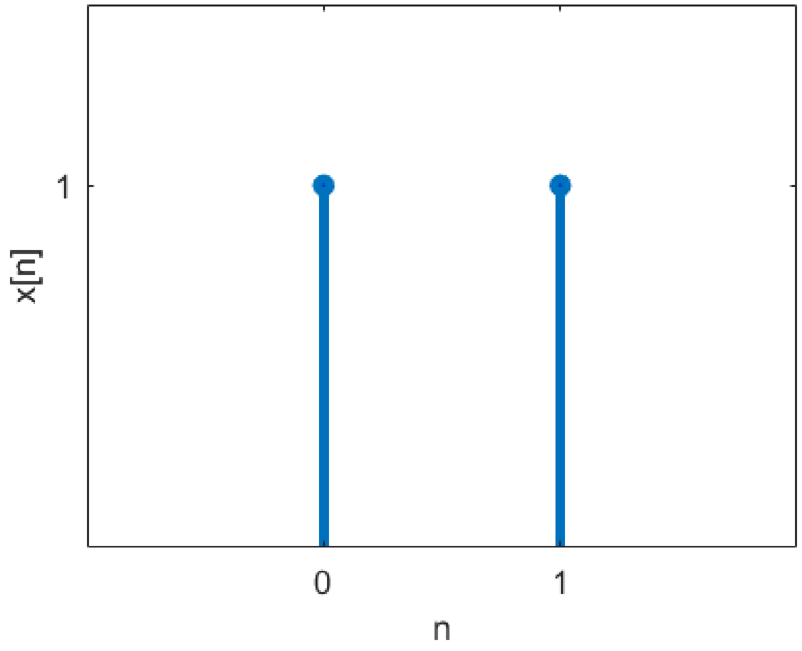
c. $x[n] = u[n] - u[n-5]$

Problem 3.5 (a) – Solution

$$x[n] = \delta[n] + \delta[n-1]$$

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 1\}$$

$$h[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 1, -1\}$$



$$x[k] = \{\underset{\substack{\uparrow \\ k=0}}{1}, 1\}$$

$$h[n-k] = \{-1, 1, \underset{\substack{\uparrow \\ k=n}}{2}\}$$

$$y[n] = x[n] * h[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 3, 0, -1\}$$

Problem 3.5 (a) – Another Solution

a. Since the system is linear

$$y[n] = \text{Sys}\{\delta[n] + \delta[n-1]\} = \text{Sys}\{\delta[n]\} + \text{Sys}\{\delta[n-1]\}$$

The system is also time-invariant, therefore

$$\text{Sys}\{\delta[n]\} = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 1, -1 \} \quad \Rightarrow \quad \text{Sys}\{\delta[n-1]\} = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 1, -1 \}$$

and

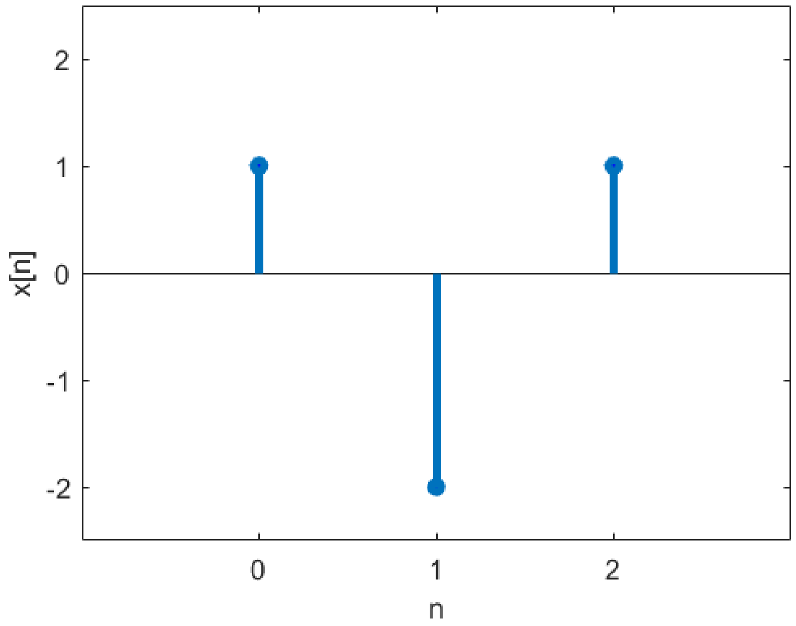
$$y[n] = \{ \underset{\substack{\uparrow \\ n=0}}{2}, 3, 0, -1 \}$$

Problem 3.5 (b) – Solution

$$x[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

$$x[n] = \{\underset{\underset{n=0}{\uparrow}}{1}, -2, 1\}$$

$$h[n] = \{\underset{\underset{n=0}{\uparrow}}{2}, 1, -1\}$$



$$x[k] = \{\underset{\underset{k=0}{\uparrow}}{1}, -2, 1\}$$

$$h[n-k] = \{-1, 1, \underset{\underset{k=n}{\uparrow}}{2}\}$$

$$y[n] = x[n] * h[n] = \{\underset{\underset{n=0}{\uparrow}}{2}, -3, -1, 3, -1\}$$

Problem 3.5 (b) – Another Solution

b. Since the system is linear

$$y[n] = \text{Sys}\{\delta[n] - 2\delta[n-1] + \delta[n-2]\}$$

$$= \text{Sys}\{\delta[n]\} - 2\,\text{Sys}\{\delta[n-1]\} + \text{Sys}\{\delta[n-2]\}$$

The system is also time-invariant, therefore

$$\text{Sys}\{\delta[n-1]\} = \{\underset{\substack{\uparrow \\ n=0}}{0}, 2, 1, -1\}$$

and

$$\text{Sys}\{\delta[n-2]\} = \{\underset{\substack{\uparrow \\ n=0}}{0}, 0, 2, 1, -1\}$$

The response is

$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, -3, -1, 3, -1\}$$

Problem 3.5 (c) – Solution

$$x[n] = u[n] - u[n-5]$$

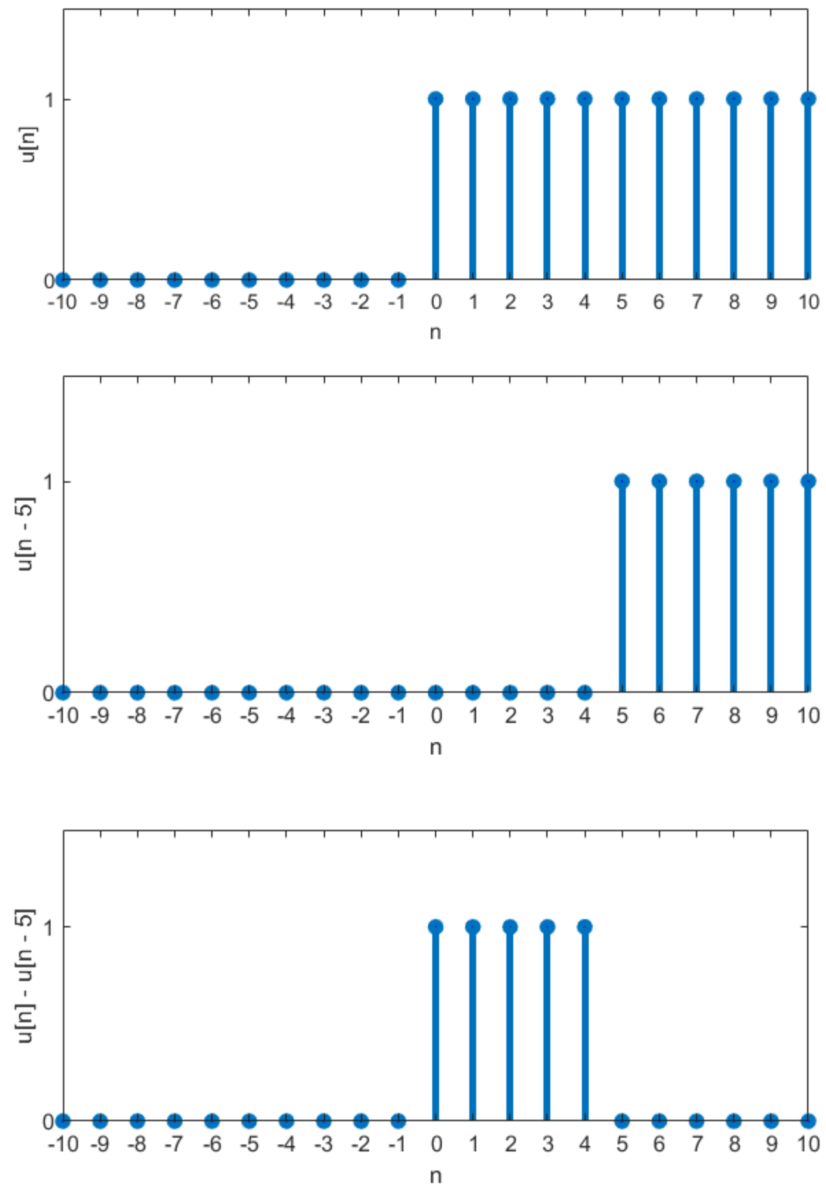
$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, 1\}$$

$$h[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 1, -1\}$$

$$x[k] = \{\underset{\substack{\uparrow \\ k=0}}{1}, 1, 1, 1, 1\}$$

$$h[n-k] = \{-1, 1, \underset{\substack{\uparrow \\ k=n}}{2}\}$$

$$y[n] = x[n] * h[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 3, 2, 2, 2, 0, -1\}$$



Problem 3.5 (c) – Another Solution

c.

$$u[n]-u[n-5]=\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]+\delta[n-4]$$

Using the linearity of the system we have

$$y[n]=\text{Sys}\{\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]+\delta[n-4]\}$$

$$=\text{Sys}\{\delta[n]\}+\text{Sys}\{\delta[n-1]\}+\text{Sys}\{\delta[n-2]\}+\text{Sys}\{\delta[n-3]\}+\text{Sys}\{\delta[n-4]\}$$

Since the system is also time-invariant, we have

$$\text{Sys}\{\delta[n-1]\}=\{\underset{\substack{\uparrow\\ n=0}}{0},2,1,-1\}$$

$$\text{Sys}\{\delta[n-2]\}=\{\underset{\substack{\uparrow\\ n=0}}{0},0,2,1,-1\}$$

$$\text{Sys}\{\delta[n-3]\}=\{\underset{\substack{\uparrow\\ n=0}}{0},0,0,2,1,-1\}$$

$$\text{Sys}\{\delta[n-4]\}=\{\underset{\substack{\uparrow\\ n=0}}{0},0,0,0,2,1,-1\}$$

The output signal is

$$y[n]=\{\underset{\substack{\uparrow\\ n=0}}{2},3,2,2,2,0,-1\}$$

Causality in Discrete-Time Systems

- A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.
- A discrete-time system defined by the relationship is causal

$$y[n] = y[n-1] + x[n] - 3x[n-1]$$

- A discrete-time system defined by the relationship is non-causal

$$y[n] = y[n-1] + x[n] - 3x[n+1]$$

Stability in Discrete-Time Systems

- A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal is guaranteed to produce a bounded output signal.
- A discrete-time input signal $x[n]$ is said to be bounded if an upper bound B_x exists such that

$$|x[n]| < B_x < \infty \quad \text{implies that} \quad |y[n]| < B_y < \infty$$

Causality and Stability in DTLTI Systems

- The impulse response of a causal DTLTI should be equal to zero for all negative index values.

$$h[n] = 0 \qquad \text{for all } n < 0$$

- For a DTLTI system to be stable, its impulse response must be absolute summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Problem 3.25

3.25. Determine whether the system is causal and/or stable.

a. $$y[n] = \text{Sys}\{x[n]\} = \sum_{k=-\infty}^{n} x[k]$$

c. $$y[n] = \text{Sys}\{x[n]\} = \sum_{k=0}^{n} x[k] \qquad \text{for } n \geq 0$$

e. $$y[n] = \text{Sys}\{x[n]\} = \sum_{k=n-10}^{n+10} x[k]$$

Problem 3.25 (a) – Solution

a. $$y[n] = \text{Sys}\{x[n]\} = \sum_{k=-\infty}^{n} x[k]$$

$$\text{Sys}\{\delta[n]\} = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Therefore, $h[n] = u[n]$.

Since $h[n] = 0$ for all $n < 0$, the system is causal.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} 1 \to \infty$$

So, the system is not stable.

Problem 3.25 (c) – Solution

c. $$y[n] = \text{Sys}\{x[n]\} = \sum_{k=0}^{n} x[k] \qquad \text{for } \; n \geq 0$$

$$\text{Sys}\{\delta[n]\} = \sum_{k=0}^{n} \delta[k] = 1 \;\text{ for }\; n \geq 0$$

Therefore, $h[n] = u[n]$.

Since $h[n] = 0$ for all $n < 0$, the system is causal.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} 1 \rightarrow \infty$$

So, the system is not stable.

Problem 3.25 (e) – Solution

e. $$y[n] = \text{Sys}\{x[n]\} = \sum_{k=n-10}^{n+10} x[k]$$

$$\text{Sys}\{\delta[n]\} = \sum_{k=n-10}^{n+10} \delta[k] = \begin{cases} 1, & n-10 \le 0 \le n+10 \quad \Rightarrow \quad -10 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$$

Therefore $h[n] = u[n+10] - u[n-11]$.

Since $h[n] \neq 0$ for all $n < 0$, the system is not causal.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-10}^{10} 1 < \infty$$

So, the system is stable.